Topology and its Applications 22 (1986) 301-313
North-Holland

TOPOLOGICAL REALIZATIONS OF CHAIN COMPLEXES 1. THE GENERAL THEORY

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Received 12 February 1985
Revised 13 June 1985

This paper studies the following question: Given a group π, and a projective $\mathbb{Z}\pi$-chain complex C, does there exist a topological space with a fundamental group π and with the property that the chain-complex of its universal cover is chain-homotopy equivalent to C? This is a generalization of the Steenrod Problem. In the Steenrod Problem (proposed by Steenrod in 1960) the chain complex was a projective resolution of a $\mathbb{Z}\pi$-module. The present paper develops an obstruction theory for the existence of topological realizations of a chain-complex, algebraically classifies these realizations (if the obstructions vanish), and proves that rational chain-complexes are always stably realizable.

AMS(MOS) Subj. Class: Primary 55S45; Secondary 55M99

Steenrod problem   Equivariant Moore spaces

Introduction

The Steenrod Problem was studied by a number of people (including the author of the present paper) and a number of different approaches were developed. The first example of a module that wasn't topologically realizable (at least the way the problem was stated above) was due to Gunnar Carlsson in [3]. Several obstruction theories were also developed to topologically realizing a module—see [13, 14, 1, 10 and 11]—and several other counterexamples were discovered.

The more general question of the topological realizability of *chain complexes* is interesting in connection with the question of which homotopy types of manifolds exist and what group-actions can be imposed on manifolds. In many interesting cases one can describe the equivariant chain-complex that a manifold with a prescribed group-action would have. At that point the question arises of whether there exists a *topological space* realizing the given chain-complex (if the space exists various forms of surgery theory can be used to study the question of the existence of the manifold).

This more general question has remained open, however. The theories developed to study the Steenrod Problem haven't been much help in resolving this issue, either, except in fairly trivial cases.

The present paper attempts to build a *Postnikov tower* whose chain complex is chain-homotopy equivalent to the given complex. The theory of Wall in [15] can then be used to find a *CW*-complex whose cellular chain complex is *isomorphic* to the complex that we started with.

The approach in the present paper may be regarded as an *extension of a dual* to the theory in [13]. In [13] the chain complex of a partial Postnikov tower was mapped to the projective resolution that was being topologically realized. In the present paper the chain complex in question is mapped to that of the partial Postnikov tower.

The main results are as follows:

Theorem. *Given a $\mathbb{Z}\pi$-chain complex T such that $H_0(T)=\mathbb{Z}$ and $H_1(T)=0$ there is an obstruction theory for determining whether T is chain-homotopy equivalent to the chain-complex of a topological space. The obstructions are elements of $H^i(T;\, M_i)$ where the M_i are $\mathbb{Z}\pi$-modules computed inductively during the construction of a topological realization of T. In particular, if T is finite dimensional, there are only a finite number of nontrivial obstructions to realizing T.*

Theorem. *If T is a finite-dimensional rational chain-complex such that $H_0(T)=0$ (i.e. a projective chain-complex over $\mathbb{Q}\pi$) then there exists a positive integer n such that $Q\oplus_\eta \sum^n T$ is topologically realizable. Here Q is a free $\mathbb{Q}\pi$-resolution of $\mathbb{Q}$ and $Q\oplus_\eta \sum^n T$ is any twisted direct sum.*

Remark. A twisted direct sum $Q\oplus_\eta \sum^n T$ is defined to be $\sum^{-1}\mathfrak{A}(\eta)$, where $\mathfrak{A}(*)$ is the algebraic mapping cone and $\eta: Q\to\sum^{n+1} T$ is a chain-map. This construction is introduced here because we want to suspend T without killing the $\mathbb{Q}$ is dimension 0 that means it is a *connected space.* In this theorem T represents the kernel of the augmentation and the twisted equivalent to T—the twisting map will represent the first k-invariant. See [12] for more information on the twisted direct sum.

Even in the case of equivariant Moore spaces the present approach seems to make computation of the obstructions easier to carry out (although it is shown that the present approach is *equivalent* to the theory in [13] for equivariant Moore spaces).

The organization of the paper is as follows:

Section 1 describes the obstruction theory in terms of an algebraic construction and a topological one. It also defines the obstructions to carrying out the algebraic step and proves the results stated above.

Section 2 proves that the topological step in the construction of the realization of T can always be carried out.

1. The obstruction theory

In this section we will develop the main geometric construction that is used to realize equivariant chain-complexes. Throughout this paper T will denote a $\mathbb{Z}\pi$-chain complex that we want to realize and $(T)^k$ will denote its k-skeleton. We assume that $H_0(T)=\mathbb{Z}$ and $H_1(T)=0$. All spaces will be assumed to be semi-simplicial sets and their chain-complexes will be the normalized semi-simplicial chain-complexes of their universal cover; equipped with a $\mathbb{Z}\pi$-module structure. The equivariant, semi-simplicial *chain-complex* of X will be denoted $C^\pi(X)$.

Definition 1.1. A *topological realization* of T will be defined to be a triple (i, f, X), where i is an isomorphism $i: \pi \to \pi_1(X)$ and $f: T \to C^\pi(X)$ is a chain-homotopy equivalence. Two such realizations (i_j, f_j, X_j), $j=1, 2$, will be called *equivalent* if there exists a homotopy equivalence of spaces $g: X_1 \to X_2$ such that $g^\# \circ i_1 = i_2$ and $g^\# \circ f_1$ is chain-homotopic to f_2.

The construction of a topological realization of T will be done in stages and the result of the ith stage will be a topological space denoted X_i.

Definition 1.2. If $f: C \to D$ is a chain-map of chain-complexes, $\mathfrak{A}(f)$ will denote the *algebraic mapping cone* of f. $\mathfrak{A}(f)_i = C_i \oplus D_{i-1}$, with boundary maps ∂_i given by:

$$\begin{bmatrix} d_c & 0 \\ (-1)^i & d_n \end{bmatrix}: C_i \oplus D_{i-1} \to C_{i-1} \oplus D_{i-2}.$$

Remark 1. We have the well-known exact sequence:

$$\cdots \to H_i(C) \to H_i(D) \to H_i(\mathfrak{A}(f)) \to H_{i-1}(C) \to \cdots$$

and it is well-known that $\mathfrak{A}(f)$ measures the extent to which f *fails* to be a chain homotopy equivalence.

Remark 2. Essentially, the procedure for realizing a chain-complex presented in this section is a modification of a relative version of that given in [13] for constructing equivariant Moore spaces. In the present paper we will build a Postnikov tower whose chain complex is equivalent to T. This is done by forming fibrations (with fiber a suitable Eilenberg–MacLane space) that have the effect of killing homology modules of $\mathfrak{A}(f)$, where f is a map from T to the chain complex of the space constructed so far.

The construction begins with $(T)^2$ as the 2-skeleton of T and $X_0 = X_2 = K(\pi, 1)$. Let $C^\pi(X_0) = Z_+$, which is a $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$ and let $f_2: (T)^2 \to C^\pi(X_2)$ be the unique chain-homotopy class of chain maps that induces an isomorphism of H_0—such a map exists because $H_0(T)=\mathbb{Z}$. Then $X_2 = K(\pi, 1)$ is the 2-dimensional approximation to a topological realization X of T and f_2 is the 2-dimensional approximation to a map f from T to the chain complex of X that is a chain-homotopy equivalence. In general $f_k : (T)^k \to C^\pi(X_k)$ is a chain-map such that $\mathfrak{A}(f_k)$ is acyclic in dimensions $<k$. The construction of X_{k+1} from X_k proceeds as follows:

Step A_k. Extend f_k to $(T)^{k+1}$ forming g_{k+1}. Strictly speaking, this is not an extension since f_k may be modified in dimension k in the process. We require that $g_{k+1} | (T)^{k+1} = f_k | (T)^{k-1}$.

Step B_k. Form a fibration over X_k in such a way that $H_k(\mathfrak{A}(g_{k+1}))$ is killed. The fiber of this fibration is a $K(H_k(\mathfrak{A}(g_{k+1})), k-1)$ and the total space is X_{k+1}) and this lift is the map f_{k+1}.

Remark. The remainder of this section will be spent giving the details of these steps, showing that they result in a topological realization of T and showing that the obstructions to carrying out these steps vanish identically if T is topologically realizable.

Definition 1.3. Let f_k, $(T)^{k+1}$ be as in step k described above. Define a class (which will be called the kth *obstruction to realizing* T) $c_k \in H^{k+1}(T; H_k(C^\pi(X_k)))$ as follows:

(a) Note that, since f_k is a chain map, the *cycle submodule of* T_k is mapped into the cycle submodule of $C^\pi(X_k)_k$ so that we get a map from the cycle submodule of T_k to $H_k(C^\pi(X_k))$.

(b) Consider the composite $T_{k+1} \to Z(T_k) \to H_k(C^\pi(X_k))$, where the map on the left is the boundary map of T and that on the right is induced by f_k. This composite defines a cocycle that gives the class c_k.

Remark 1. It turns out that this is the only place where nontrivial obstructions to realizing t will be encountered.

Remark 2. Suppose T is of the form $Z_+ \oplus \sum^n P$, where Z_+ is a $\mathbb{Z}\pi$-projective resolution of $\mathbb{Z}$ and P is a projective resolution of a module M. A topological realization of this complex constitutes an equivalent Moore space of type (M, n, π).

The problem of realizing such complexes was studied in [13] and an obstruction theory was developed with obstructions that were essentially homological k-invariants of the complexes $C^\pi(X_k)$. The obstructions in the present paper turn out to also be homological k-invariants, in this case. If the first homological k-invariant of $C^\pi(X_k)$ is some class $\alpha \in \mathrm{Ext}^{k+1}_{\mathbb{Z}\pi}(M, H_{n+k})$ then $C^\pi(X_k)$ is chain-homotopy equivalent (at least up to dimension $n+k+1$) to a twisted direct sum $Z_+ \oplus (\sum^n P \oplus_\alpha \sum^{n+k} H)$ where a cocycle representative for α is used to define a chain map $\alpha : \sum^n P \to \sum^{n+k+1} H$ and a twisted direct sum is a *desuspension* of an algebraic mapping cone (see [9] for details). If we define $f_k : (Z_+ \oplus \sum^n P)^{n+k} \to Z_+ \oplus_\alpha \sum^{n+k} H)$ to be the inclusion it is not hard to see that the obstruction, in the sense of the present paper, is precisely α.

In general we will have to take into account the maps in homology induced by the homotopy equivalence with the twisted direct sum. It is clear, however, that the theory presented in the present paper is equivalent to that in [13] in the case of equivariant Moore spaces.

Proposition 1.4. *Step A_k can be carried out if and only if the class $c_k \in H^{k+1}(T; H_k(C^\pi(X_k)))$, defined above, vanishes.*

Proof. (1) Suppose the class vanishes. Then the cocycle defined by the composite $d_{k+1} \circ f_k$ is the pullback over d_{k+1} of a map $g: T_k \to H_k(C^\pi(X_k))$—i.e. the map from $d_{k+1}(T_{k+1})$ to the cycle submodule of $C^\pi(X_k)_k$ defining c_k extends to all of T_k. Call the extension $g: T_k \to$ (cycle submodule of ${}^\pi(X_k)_k$). Now replace f_k by $g_{k+1} = f_k - g$: $(T)^k \to C^\pi(X_k)$ (where we only alter f_k in dimension k). The result is still a chain map since the image of g is the cycle submodule of $C^\pi(X_k)_k$, and it agrees with f_k on $(T)^{k+1}$. The result is a map $g_{k+1}: (T)^k \to H_k(C^\pi(X_k))$ whose restriction to $d_{k+1}(T_{k+1})$ vanishes. This vanishing implies that g_{k+1} lifts to a map $T_k \to$ (cycle submodule of $C^\pi(X_k)_k$) with the property that the image of $d_{k+1}(T_{k+1})$ is in the *boundary submodule.* Since T_{k+1} is projective we can clearly lift $g_{k+1} \circ d_{k+1}: T_{k+1} \to C^\pi(X_k)_k$ to get a map $g_{k+1}: T_{k+1} \to C^\pi(X_k)_{k+1}$ and the result is clearly a *chain-map.*

(2) Suppose the map f_k extends to a map $g_{k+1}: (T)^{k+1} \to C^\pi(X_k)$. Notice that the class c_k is linear with respect to f_k in the following sense: if f'_k and f''_k are two maps from $(T)^k \to C^\pi(X_k)$ then $c_k(f'_k + f''_k) = c_k(f'_k) + c_k(f''_k)$. Now note that $c_k(g_{k+1})$ vanishes identically (since g_{k+1} is a chain map in dimension $k+1$ so that the image of $d_{k+1}(T_{k+1})$ is in the boundary submodule of $C^\pi(X_k)_k$). Also note that $g_{k+1} - f_k \mid T_k$ must have its image in the cycle submodule of $C^\pi(X_k)_k$ since it is part of a chain map that vanishes in lower dimensions (because the extension must agree with the original map in lower dimensions). It follows that $c_k(g_{k+1} - f_k) = 0$—in this case the cocyle doesn't vanish identically but it is a coboundary. The conclusions follows. □

Remark 1. Varying g_{k+1} by a coboundary amounts to a chain homotopy and so replaces $\mathfrak{A}(g_{k+1})$ by an *isomorphic complex.* Adding an element of $H^k(T; H_k(C^\pi(X_k)))$ does alter $\mathfrak{A}(g_{k+1})$ but only in dimensions $\geq k$. In particular it doesn't change the fact that g_{k+1} is $k-1$-connected.

Remark 2. Note that in argument (1) we could have added any cocycle $y_k \in H^k(T; H_k(C^\pi(X_k)))$ to g to get the map g_{k+1}. Furthermore, if g' is any other extension of f_k the difference between g_{k+1} and g' (in dimension k) is a cocycle in $H^k(T; H_k(C^\pi(X_k)))$. If $e: H^k(T; H_k(C^\pi(X_k))) \to \mathrm{Hom}_{Z\pi}(H_k(T), H_k(C^\pi(X_k)))$ is the evaluation map then adding y_k to g alters the resulting map of homology modules by $e(y_k)$. They also determine higher c_k's so that the obstructions to realizing T might vanish for some choices of y_k but not for others.

Remark 3. In general f_k defines an element $s_k \in H^k(T; H_k(C^\pi(X_k)))$ that will be called the kth *s-invariant* of the realization. These invariants classify the realizations up to equivalence—see Definition 1.1.

Section 2 of this paper will be spent proving the following:

Proposition 1.5. *Step B_k can always be carried out.*

The main result of this paper is:

Theorem 1.6. *If T is a finite-dimensional chain-complex with top dimension n then T can be topologically realized if and only if there exists a sequence of choices of $y_k \in H^k(T; H_k(C^\pi(X_k)))$ for $k < n$ such that for each k, c_{k+1} vanishes. If T is infinite-dimensional the same result is true but the c_k must vanish for all values of k.*

Remark 1. The results of Wall in [15] imply that if T is realizable in the sense of this paper then we can even find a *CW complex* realizing T whose *cellular chain complex* is isomorphic to T.

Remark 2. Not all elements of $H^k(T; H_k(C^\pi(X_k)))$ can occur as obstructions to some realization problem. The following example (which I studied in detail in [14]) demonstrates this: In that example I constructed a twisted direct sum $Z \oplus_\xi \sum^2 Z$, where the fundamental group was $\mathbb{Z}^5$ and $\xi \in H^3(\mathbb{Z}^5, \mathbb{Z})$. It turned out that the first obstruction to realizing the twisted direct sum was precisely 2ξ so that obstruction elements are multiples of 2. This must be true whenever $\pi = \mathbb{Z}^5$ and $H_2(T) = \mathbb{Z}$, since the 3-skeleton of T will be equivalent to a twisted direct sum like the one in the example.

Proof. (1) Suppose that the hypothesis is satisfied.

Claim. There exists a chain-map $w_k : T \to C^\pi(X_k)$ for all k, and this map commutes with the chain-maps $p_{k+1} : C^\pi(X_{k+1}) \to C^\pi(X_k)$ induced by the projections of the fibrations, for all k.

This is an immediate consequence of the way the maps $f_k : (T)^k \to C^\pi(X_k)$ were defined. Since $g_{k+1} | (T)^{k-1} = f_k | (T)^{k-1}$ and f_{k+1} is a lift of g_{k+1} we can define $w_k : T_n \to C^\pi(X_k)_n$ to be the composite $P_k \circ \cdots \circ P_{n+2} \circ f_{n+2}$ when $n > k-2$ and f_k otherwise. So we have a tower of fibrations $\cdots X_k \to X_{k-1} \to \cdots \to X_2 \to X_0$ such that there exists a chain map from T to each term of the induced inverse system of chain-complexes. This gives rise to a chain map from T to the chain complex of the inverse limit $X_\leftarrow$ that is a chain homotopy equivalence.

(2) Conversely, if T is topologically realizable by a space X then there clearly exists a chain map from T to the chain-complexes of all the partial Postnikov towers of X so that Proposition 1.4 implies the conclusion. □

The remainder of this section will be spent exploring consequences of this result. Note that, since the first nontrivial obstruction is c_2 and since $c_k \in H^{k+1}(T; H_k(C^\pi(X_k)))$:

Proposition 1.7. *If the top dimension of T is n there are at most $n-2$ nontrivial obstructions to realizing T topographically.*

Now we will consider rational chain-complexes. These are projective chain complexes over $\mathbb{Q}\pi$ rather than $\mathbb{Z}\pi$.

Definnition 1.8. A *stable realization* of a chain complex T, with $H_0(T)=0$ is a realization of $Z \oplus_\eta \sum^m T$, for some value of m and some twisting map $\eta : Z \to \sum^{m+1} T$, where Z is a free $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$.

Lemma 1.9. *If T is a finite-dimensional rational chain-complex, then T is stably realizable.*

Remark 1. A similar result is true for infinite-dimensional rational chain-complexes if we work in categories of stable chain-complexes and spaces. Roughly speaking a topological realization of a chain-complex in such a category is a sequence of stable realizations of all finite skeleta.

Remark 2. The corresponding unstable result is not true—the example given in [14] is also a rational non-realizable chain-complex.

Proof. Suppose T is n-dimensional. We will find a realization of $\sum^{n+1} T$. We will use the fact that the homology of a rational Eilenberg–MacLane space vanishes in the stable range—i.e. $K(M, n+1)$ has vanishing homology in dimensions $>n+1$ and $<2n+2$—see [6].

A simple inductive argument (using the Serre exact sequence of a fibration, for instance) shows that $H_k(X_k)=0$ for $n+1<k<2n+2$. In other words, whenever we want to adjoin a new term to the partial Postnikov tower the homology module in the dimension that concerns us starts out being zero. But this immediately implies that the obstruction to adjoining the new term vanishes identically.

Furthermore, above dimension $2n+1$ the chain-modules of T vanish so all obstructions vanish. □

2. Proof of Proposition 1.4

We will begin by developing some of the algebraic machinery needed to do computations with DGA-algebras. We will also present some of the relevant results of Gugenheim on computing chain complexes of fibrations (see [8]).

Definition 2.1. Let $f: C \to D$, $g: D \to C$ be maps of chain-complexes. Then:

(1) if f maps each C_i to D_{i+k} then f will be called a *map of degree k*;

(2) if f is a map of degree k then df is defined to be $d_D \circ f + (-1)^{k+1} f \circ d_c$. The map f is defined to be a *chain map* if it is of degree 0 and $df = 0$.

(3) if f and g, above, are both chain maps and:

(a) $f \circ g = 1_D$, and $g \circ f = d\varphi$, where φ is some map of degree $+1$; and

(b) $f \circ \varphi = 0$, $\varphi \circ g = 0$, and $\varphi^2 = 0$;

then the triple (f, g, φ) is called a *contraction of C onto D*. The map f is called the *projection* of the contraction, and g is called the *injection*.

Remark 1. Since df has the special meaning given above, we will follow Gugenheim in [7] in using $d \circ f$ to denote the *composite*.

Remark 2. We will also use the convention that if $f: C_1 \to D_1$, $g: C_2 \to D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogeneous element), then $(f \otimes g)(a \otimes b) = (-1)^{\deg(g)\deg(a)} f(a) \otimes g(b)$. This convention simplifies some of the common expressions in homological algebra. For instance the differential, $d_\otimes$, of the tensor product $C \otimes D$ is just $d_c \otimes 1 + 1 \otimes d_D$.

Remark 3. It is not difficult to see that the definition of a chain-map given above coincides with the usual definition.

Remark 4. The definition of a contraction of chain complexes given here is slightly stronger than the original definition due to Eilenberg and MacLane in [5], since they don't require the chain-homotopy to be self-annihilating.

The present definition is due to Weishu Shih in [12].

Definition 2.2. (1) A triple (A, φ, η) will be called a DGA-algebra if A is a $\mathbb{Z}\pi$-chain complex and φ and η are $\mathbb{Z}\pi$-chain maps: $\varphi: A \otimes A \to A$, $\eta: \mathbb{Z} \to A$ such that $\varphi \circ (\eta \otimes 1_A) = \varphi \circ (1_A \otimes \eta) = 1_A$ and $\varphi \circ (1_A \otimes \varphi)$;

(2) A triple (B, ψ, ε) will be called a DGA-algebra if B is a $\mathbb{Z}\pi$-chain complex and ψ and ε are $\mathbb{Z}\pi$-chain maps: $\psi: B \to B \otimes B$, $\varepsilon: B \to \mathbb{Z}$ such that $(\varepsilon \otimes 1_B) \circ \psi = (1_B \otimes \varepsilon) \circ \psi = 1_B$ and $(\psi \otimes 1_B) \circ \psi = (1_B \otimes \psi) \circ \psi$;

Remark 1. The chain complex of any topological space X can be regarded as a DGA coalgebra via the Eilenberg-Zilber theorem applied to the diagonal map of the space.

Remark 2. Using the definition of Eilenberg-MacLane spaces given in [5], the chain complex of any Eilenberg-MacLane space is a DGA algebra.

Definition 2.3. Let B be a DGA-coalgebra and A be a DGA-algebra. Let x and y be chain maps from B to A. Then:

(1) the *cap product with respect to* x, denoted $x\cap$, is defined to be the composite $(1_B\otimes\varphi)\circ(1_B\otimes x\otimes 1_A)\circ(\psi\otimes 1_A): B\otimes A\to B\otimes A$;

(2) the *cup product of* x *and* y, denoted $x\cup y$, is the composite $\varphi\circ(x\otimes y)\circ\psi: B\to A$;

(3) if x is a map of degree -1 that has the property that $dx+x\cup x=0$ then the *twisted tensor product* $B\otimes_x A$ is defined to be the chain-complex $B\otimes A$, equipped with the differential $\mathrm{d}_x=\mathrm{d}_{B\otimes A}+x\cap$.

Remark 1. The condition on x in statement 3 implies that the differential for the twisted tensor product is self-annihilating—see [7]. In this case the map x is called the *twisting cochain* of the twisted tensor product.

Remark 2. Twisted tensor products were originally defined to study the chain complex of a fibration. Fibrations can be described as semi-simplicial complexes as 'twisted cartesian products'—see [8, p. 405] or [12, Chapter 2]. The main result (see [8] for details) in this direction is that there exists a *contraction* $(f_\xi, g_\xi, \varphi_\xi)$: $C(B\times_\xi F)\to C(B)\otimes_\xi C(F)$ from the chain complex of the total space of a fibration to the twisted tensor product of that of the base and that of the fiber.

Since all maps involved are natural the corresponding statement is also true for the equivariant chain-complexes. This is significant for the chain-complex of the fiber even though it will be simply-connected because it will be equipped with an action of $\pi_1(B)$. The twisted tensor product is equipped with the diagonal π-action.

The twisting cochain in the twisted tensor product is related to the twisting function by a formula given in [8]—essentially, if w is the twisting function the twisting cochain is a polynomial function of $w-1$ (where F is regarded as a topological ring so its simplices can be multiplied).

Furthermore, we can assume that all chain-complexes are normalized—see [5, sections 4, 5], which defines normalization and proves the Eilenberg–Zilber theorem for such complexes.

This assumption will be in effect throughout the remainder of this section.

Remark 3. Suppose that the normalized chain complex of F vanishes below dimension n (except for a copy of $\mathbb{Z}$ in dimension 0) and that the 2-skeleton of B is the same as that of Z_+, a $\mathbb{Z}\pi$-free resolution of $\mathbb{Z}$. Then the twisted tensor product will consist of a (twisted) direct sum of:

(a) a copy of $(Z_+)^2\otimes C(F)^+=C^\pi(F)''$;

(b) a copy of $C^\pi(B)\otimes\mathbb{Z}=C^\pi(B)$;

(c) a copy of $\sum^n C^\pi(B)^+\otimes(\sum^2 C^\pi(F)^+)$;

Here the $+$ denotes the kernel of the augmentation map and we have identified the 2-skeleton of $C^\pi(B)$ with that of Z_+. Thus, below dimension $n+2$ the twisted tensor product is essentially a twisted direct sum or Σ^{-1} of the algebraic mapping cone of the map $\xi: C^\pi(B)\to\sum C^\pi(F)''$. In this dimension range ($<n+3$) the identity that the twisting cochain must satisfy is essentially that of a chain-map to the suspension.

Here the differential on the twisted tensor product is such that $C(F)$ (inclusion of the fiber) and the twisted direct sum $C^{\pi}(B) \oplus_{\xi} C^{\pi}(F)''$ are subcomplexes of $C^{\pi}p(B) \otimes_{\xi} C(F)$.

Remark 4. Suppose X is a topological space acted upon by a group π and $f: X \to K(M, n)$ is a π-equivariant map, where M is a $\mathbb{Z}\pi$-module. Then it is possible to pull back the universal fibration over $K(M, n)$ which has fiber a $K(M, n-1)$, a contractible total space, and is also acted upon by π. The result is a $K((M, n-1)$-fibration over X with π acting on X, the total space, and the fiber and such that the projection is π-equivalent.

Now we are in a position to describe the fibration that must be used to construct X_{k+1}. We assume given a map $g_{k+1}: (T)^{k+1} \to C^{\pi}(X_k)$ such that $H_i(\mathfrak{A}(g_{k+1})) = 0$ for $i < k$. Consider the homomorphism

$$h^*: H^+(\mathfrak{A}(g_{k+1}); H_k(\mathfrak{A}(g_{k+1}))) \to H^+(X_k; H_k(\mathfrak{A}(g_{k+1})))$$

induced by the inclusion $h: C^{\pi}(X_k) \to \mathfrak{A}(g_{k+1})$. Since $\mathfrak{A}(g_{k+1})$ is a projective $\mathbb{Z}\pi$-chain complex that is bounded from below it follows that the evaluation map

$$e: H^k(\mathfrak{A}(g_{k+1}); H_k(\mathfrak{A}(g_{k+1}))) \to \mathrm{Hom}_{\mathbb{Z}\pi}(H_k(\mathfrak{A}(g_{k+1})), H_k(\mathfrak{A}(g_{k+1})))$$

is an isomorphism. Select a class $c \in H^k(\mathfrak{A}(g_{k+1}); H_k(\mathfrak{A}(g_{k+1})))$ that maps to the identity map of $H_k(\mathfrak{A}(g_{k+1}))$ and form the fibration over X_k classified by $h^*(c)$—this will have fiber a $K(H_k(\mathfrak{A}(g_{k+1})), k-1)$.

This is done by regarding $h^*(c)$ as a map $X_k \to K(H_k(\mathfrak{A}(g_{k+1})), k)$ and pulling back the universaal fibration of the target over this map. If we use the semi-simplicial complex for $K(H_k(\mathfrak{A}(g_{k+1})), k)$ given in [5, section 17] this map can be described very explicitly.

Simply map a *k-dimensional simplex*, s, of X_k to the unique simplex of the Eilenberg-MacLane space whose symbol is $[\sigma(s)] \otimes 1 \otimes \cdots \otimes 1$ (see [5] or [4, p. 13]) where $\sigma: C^{\pi}(X_k)_k \to H_k(\mathfrak{A}(g_{k+1}))$ is a cocycle representing the class c; and then extend the (geometric) map to all of X_k. The extension will be unique—see [4, p. 14].

Let $\tau: X_{k+1} \to X_k$ be the pullback of the universal fibration over $K(H_k(\mathfrak{A}(g_{k+1})), k)$.

Proposition 2.4. *The chain map* $g_{k+1}: (T)^{k+1} \to C^{\pi}(X_k)$ *can be lifted to a chain map* $f_{k+1}: (T)^{k+1} \to C^{\pi}(X_{k+1})$. *This lift is unique, up to a chain-homotopy.*

Proof. We will make use of the contraction mentioned above:

$$(r_{\tau}, s_{\tau}, \varphi_{\tau}): C^{\pi}(X_{k+1}) \to C^{\pi}(X_k) \otimes_{\tau} C^{\pi}(K(H_k(\mathfrak{A}(g_{k+1})), k-1)).$$

We will really map $(T)^{k+1}$ to the twisted tensor product via a map f'_{k+1} and compose this map with s_{τ} to get the map f_{k+1} to $C^{\pi}(X_{k+1})$.

Claim. It is enough to lift g_{k+1} to the twisted direct sum $C^{\pi}(X_k) \oplus_{\tau} C^{\pi}(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''$. This follows from Remark 3 following 2.3, which implies that this twisted direct sum is a subcomplex of the twisted tensor product.

Since $C^\pi(K(H_k(\mathfrak{A}(g_{k+1})), k-1)''$ vanishes below dimension $k-1$ we can define f'_{k+1} to equal g_{k+1} in this range—i.e. to have its target in the $C^\pi(X_k)$-summand. It is, therefore, only necessary to define f'_{k+1} in dimensions k and $k+1$. Consider the map $\tau\circ g_{k+1}: T_k \to C^\pi(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''_k$. Since it vanishes identically below dimension k, and since it induces the zero map in homology, it follows that there exists some map $\rho: T_k \to C^\pi(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''_{k+1}$ such that $\tau\circ g_{k+1} = d\circ\rho$. So, define $f'_{k+1} = g_{k+1}\oplus\rho: T_k \to C^\pi(X_k)\oplus_\tau C^\pi(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''$. We can also extend f'_{k+1} to dimension $k+1$ since $H_k(K(H_k(\mathfrak{A}(g_{k+1})), k-1)) = 0$ (see [6, section 20]).

Now we will show that the map $f'_{k+1}: T \to C^\pi(X_k)\oplus_\tau C(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''$ actually lifts g_{k+1}. It is not hard to see that the composite of f'_{k+1} with the projection to the first factor coincides with g_{k+1}. It follows that the corresponding map $f'_{k+1}: T \to C^\pi(X_k)\oplus_\tau C(K(H_k(\mathfrak{A}(g_{k+1})), k-1)$ composed with $1\otimes\varepsilon$ also coincides with g_{k+1}, where $\varepsilon: C(K(H_k(\mathfrak{A}(g_{k+1})), k-1)) \to \mathbb{Z}$ is the augmentation (a homomorphism of DGA-algebras that is a left inverse for the unit). That the composite $f_{k+1} = g_\tau\circ f'_{k+1}: T \to C^\pi(X_{k+1})$ is also a lift of g_{k+1} now follows from the naturality of the contraction $(r_\tau, s_\tau, \varphi_\tau): C^\pi(X_{k+1}) \to C^\pi(X_k)\otimes_\tau C(K(H_k(\mathfrak{A}(g_{k+1})), k-1))$ with respect to maps of the fiber—i.e. map the fibration to a trivial fibration over X_k that has fiber a point—that map (of spaces) coincides with the projection of the original fibration to the base space. □

Proposition 2.5. *The lift constructed above has the property that* $H_i(\mathfrak{A}(f_{k+1})) = 0$, $i \leq k$.

Proof. First note that $H_i(\mathfrak{A}(f_{k+1})) = 0$, $i \leq k-1$—i.e. we haven't lost anything by lifting the map. This follows immediately from considering the map of the exact sequence of f_{k+1} to the exact sequence of g_{k+1} induced by the projection of the fibration $X_{k+1} \to X_k$ and the fact that the fiber is acyclic below dimension $k-1$.

Now note that, in computing $H_k(\mathfrak{A}(f_{k+1}))$ we can substitute f'_{k+1} for f_{k+1} because of the existence of a contraction from $C^\pi(X_{k+1})$ onto $C^\pi(X_k)\otimes_\tau C(K(H_k(\mathfrak{A}(g_{k+1})), k-1))$—see [12]. Since the homology module we are interested in is in dimension k we are also free to substitute $H_k(\mathfrak{A}(f''_{k+1}))$ where f''_{k+1} is just f'_{k+1}, regarded as a map to the subcomplex $C^\pi(X_k)\oplus_\tau C^\pi(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''$—i.e. the rest of the twisted tensor product will have boundaries in the kth chain module so the kth homology of the whole twisted tensor product will be a quotient of the kth homology module of the twisted direct sum. We will prove that $H_k(\mathfrak{A}(f''_{k+1})) = 0$ and that will imply the conclusion.

Note that, in the dimension range that interests us (i.e. dimensions $\leq k+1$), $C^\pi(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''$ is just the $k-1$ fold suspension of a projective resolution of $H_k(\mathfrak{A}(g_{k+1}))$. Now define $\mathfrak{A}''$ to be the same as $\sum^{-1}\mathfrak{A}(g_{k+1})$ in dimensions $\leq k+1$ and acyclic in dimensions $>k$.

Now consider the map $\tau: C^{\pi}(X_k) \to C(K(H_k(\mathfrak{A}(g_{k+1})), k-1))$—this map is of degree -1 but otherwise behaves like a chain-map in dimensions $\leqslant k+1$. It is the pullback of the k-dimensional cohomology class of $\mathfrak{A}(g_{k+1})$ that evaluated to an isomorphism in homology (with coefficients in the homology module in dimension k). In other words, it is the pullback of a cochain on $\mathfrak{A}(g_{k+1})_k$ that maps each element in the cycle submodule (which is a direct summand) to its image in the homology module. This argument implies that the following diagram commutes:

$$\begin{array}{ccc} C^{\pi}(X_k)_i \to T_{i-1} \oplus C^{\pi}(X_k)_i = \mathfrak{A}(g_{k+1})_i & & \\ \downarrow \qquad\qquad\qquad \downarrow & & i=k,\ k+1 \\ C^{\pi}(X_k)_i \to C^{\pi}(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''_{i-1} & & \end{array}$$

where the upper row is the inclusion, the lower row is τ, the map $\mathfrak{A}(g_{k+1})_i \to C^{\pi}(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''_{i-1}$ is the (unique, up to a chain-homotopy) chain map (of degree -1) that maps elements of the cycle submodule of $\mathfrak{A}(g_{k+1})_k$ to [image-in-homology]$\otimes 1 \otimes \cdots \otimes 1$ in $C^{\pi}(K(H_k(\mathfrak{A}(g_{k+1})), k-1))''_{k-1}$ and sends elements of the projective complement to 0. That map is well-defined up to dimension $k+1$, and is a chain-homotopy equivalence up to that dimension. This implies that the twisted direct sum can be replaced by (i.e. there exists a chain map from the second complex to the first that is an equivalence up to dimension $k+1$) the twisted direct sum $C^{\pi}(X_k) \oplus_{\iota} \mathfrak{A}$; where $\iota: C^{\pi}(X_k) \to \sum \mathfrak{A}'$ is the chain map defined by the inclusion of $C^{\pi}(X_k)$ in the algebraic mapping cone. Consideration of the boundary maps of the twisted direct sum and the algebraic mapping cone shows immediately that this complex is essentially a twisted direct sum with respect to a map $z: T \to A$, where A is the algebraic mapping cone of $z: C^{\pi}(X_k) \to C$; and this map is an isomorphism below dimension $k+1$. Thus there exists a map $C^{\pi}(X_k) \oplus_{\iota} \mathfrak{A}' \to T \oplus_z A \to T$ in dimensions $\leqslant k+1$ that is an equivalence. The conclusion that $H_k(\mathfrak{A}(f''_{k+1})) = 0$ follows immediately, and this implies that $H_k(\mathfrak{A}(f_{k+1})) = 0$. $\square$

Acknowledgment

I am indebted to Sylvain Cappell and Andrew Ranick for their encouragement, and to Drexel University for providing me with a research grant and other support for this work. I am also indebted to the referee for a careful reading of this paper and many helpful suggestions.

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