Topology Vol. 16, pp. 461–463. Pergamon Press, 1977. Printed in Great Britain

HOMOLOGY SURGERY THEORY AND PERFECT GROUPS

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(Received 8 December 1976)

IN THIS paper we will give an algebraic proof of the following result:

THEOREM. *Let $f: G \to Q$ be a surjective homomorphism of groups such that the Kernel is the normal closure of a finitely generated perfect group P. Then the homomorphisms $\Gamma_i^s(f) \to L_i^s(Q)$, defined in Chapter 1 of* [2], *are isomorphisms for all i.*

Here $L_i^s(Q)$ denotes the Wall surgery obstruction group of Q, defined algebraically in [5], and $\Gamma_i^s(f)$ is the homology surgery obstruction group of Cappell and Shaneson, defined algebraically in [2].

This theorem was originally proved by Hausmann in [3] for i even by performing surgery on imbedded integral homology spheres. Using a similar technique, Cappell and Shaneson proved the theorem in the odd-dimensional case. See [3] for geometric proofs.

The algebraic methods of the present paper may extend to more general results relating homology surgery groups to Wall groups.

§1. ALGEBRAIC PRELIMINARIES

Throughout this paper I will denote the right ideal of $\mathbf{Z}G$ generated by elements of the form $(p-1)$, for all $p \in P$, and K will denote the corresponding ideal generated by elements of the form $(k-1)$ for all k in $\ker f$—this will actually be two-sided since $\ker f$ is normal in G. Clearly $I \subset K$ and we have:

LEMMA 1.1. $I \otimes_{\mathbf{Z}G} \mathbf{Z}Q = 0$.

Proof. $I \otimes_{\mathbf{Z}G} \mathbf{Z}Q = I/I \cdot K$. Since $I^2 \subset I \cdot K \subset I$, and since P is perfect, $I^2 = I$ and the result follows. (See [1], p. 190—this implies that $I^2 \cap \mathbf{Z}P = I \cap \mathbf{Z}P$.)

LEMMA 1.2. *Let I_g be the right ideal of $\mathbf{Z}G$ generated by elements of the form $(gpg^{-1}-1)$ for all $p \in P$, and let J be a finite sum of ideals of the form $I_{g_1} \ldots I_{g_n}$, $g_i \in G$. Then J is a finitely generated right $\mathbf{Z}G$-module such that $J \otimes_{\mathbf{Z}G} \mathbf{Z}Q = 0$.*

Proof. First note that $I_g \cdot K = I_g$—this follows from the fact that I_g is isomorphic, as a module, to I, and from Lemma 1.1. It also follows that $I_{g_1} \ldots I_{g_n} \cdot K = I_{g_1} \ldots I_{g_n}$ and that $J \cdot K = J$ so that $J \otimes_{\mathbf{Z}G} \mathbf{Z}Q = J/J \cdot K = 0$. That J is finitely generated follows from the fact that P, and therefore, I is finitely generated.

LEMMA 1.3. *Let r be an element of K. Then*:

(1) *There exists a finitely generated right ideal $J(r)$ such that $r \in J(r)$ and $J(r) \otimes_{\mathbf{Z}G} \mathbf{Z}Q = 0$.*

(2). *There exists a kernel (see* [5], *Lemma 5.3) over $\mathbf{Z}G$, $T = (F \oplus F', \varphi, \mu)$ with canonical subkernel F, and a pre-subkernel $B(r) \subset F$ (see* [2], *Section 1.1 for a definition of this term) such that the image of $B(r)$ in $F'' = F \otimes_{\mathbf{Z}G} \mathbf{Z}Q$ is all of F'' and such that there exist elements $j \in F$, $k \in F'$ with the property that* (a) *their images are 0 in $(F \oplus F') \otimes_{\mathbf{Z}G} \mathbf{Z}Q$, and* (b) *$r = \varphi(j, k) = \mu(j + k)$, and $\varphi(t, k) = 0$ for all $t \in B(r)$.*

Remarks. The ideals $J(r)$ are used in the even-dimensional case and the pre-subkernels $B(r)$ are used in the odd-dimensional case. If we recall the formation–theoretic description of homology surgery obstruction groups (see [2], Chapter 1, and [4]) it is not hard to see that the triple $((F \oplus F', \varphi, \mu), F, B(r))$, denoted $V(r)$ throughout the rest of this paper, represents the trivial element of an odd-dinensional Γ-group (see [2], Section 1.2).

Proof. (1) This is an immediate consequence of 1.2—r is a $\mathbf{Z}$-linear combination of products of conjugates of elements in P, and is therefore contained in a finite sum of ideals of the form I_g.

(2). Since $J = J(r)$ is finite generated it follows that there exists a surjective homomorphism $F \to J$, where F is a free $\mathbf{Z}G$-module of finite rank. Call the kernel $B(r)$ and regard the map from F to J as defining a linear form ρ on F. It follows from the fact that $J \otimes_{\mathbf{Z}G} \mathbf{Z}Q = 0$ that the image of $B(r)$ in $F'' = F \otimes_{\mathbf{Z}G} \mathbf{Z}Q$ is all of F''. Let F' be a free module isomorphic to F and define a kernel structure on $F \oplus F'$ in such a way that F is a canonical subkernel (see [5], Section 5)—call the result $(F \otimes F', \varphi, \mu)$. Since φ is nonsingular, and since $\mathrm{Hom}_{\mathbf{Z}G}(F, \mathbf{Z}G)$ is the image of F' under ad_φ, it follows that there exists an element k of F' such that $\rho(x) = \varphi(x, k)$ for $x \in F$. The nonsingularity of the quadratic form, $\varphi \otimes 1$, on $(F \oplus F') \otimes_{\mathbf{Z}G} \mathbf{Z}Q$ implies that the image of k is 0. The surjectivity of ρ: $F \to J$ implies that there exists an element j such that $\varphi(j, k) = \rho(j) = r$ and we may vary j by a suitable element of $B(r)$ to make its image in $(F \oplus F') \otimes_{\mathbf{Z}G} \mathbf{Z}Q$ zero, without changing $\rho(j)$. The remaining statements follow from the properties of kernels.

§2. THE EVEN-DIMENSIONAL CASE

In this case we already know that the map $\gamma_i^s(f) \to L_i^s(Q)$ is *surjective* (see [2], Chapter 1), and we must show that its kernel vanishes. Let v be an element of $\Gamma_i^s(f)$ that maps to a kernel in $L_i^s(Q)$.

Claim. We may assume that the underlying module of v is *free* and has a basis that maps to the standard basis of the kernel in $L_i^s(Q)$. This claim follows from Lemma 1.2 in [2]), i.e., lift the standard basis of the kernel to a set of elements of the underlying module of v and map a free module to it and pull back the quadratic form.

Thus, without loss of generality, we may assume that $v = (F, \varphi, \mu)$ with F free with basis $\{x_i\}$, $1 < i \leq 2k$, and with $x_i \otimes 1$ the canonical basis of $v \otimes \mathbf{Z}Q$. If $\varphi(x_i, x_j) = 0$ and $\mu(x_i) = 0$ for $1 \leq i, j \leq k$, we could conclude that v was strongly equivalent to zero (see [2], Section 1.1 for a definition of this term) in $\Gamma_i^s(f)$ and the result would follow. We will define an inductive procedure for constructing a sequence of modules with quadratic forms, each equivalent to the previous one as elements of $\Gamma_i^s(f)$, such that the final element is strongly equivalent to zero. Consider x_1, x_2 the first two basis elements of F (we assume that $k \geq 2$). Since they map to basis elements of a standard subkernel over $\mathbf{Z}Q$, it follows that $\varphi(x_1, x_2) = r_1$, $\mu(x_1) = r_2$ and $\mu(x_2) = r_3$ with r_1, r_2, r_2 contained in the ideal K. Let $M = F \oplus J(r_1) \oplus J(r_2) \oplus J(r_3)$ and define bilinear and quadratic forms on M as follows:

$$\varphi', \ \mu'|F = \varphi, \mu \text{ respectively } \varphi', \mu'|F(r_1) \oplus F(r_2) \oplus F(r_3) = 0$$

$$\varphi'(x_i, r_j) = \delta_{ij} r_j, \text{ where } r_j \text{ is contained in } J(r_j).$$

These statements, together with the identities satisfied by the bilinear and quadratic forms of an element of $\Gamma_i^s(f)$ completely define $v' = (M, \varphi', \mu')$ (see [2], Section 1.1).

Claim. $v' = v$ in $\Gamma_i^s(f)$. Form $(F, \varphi, \mu) \oplus (M, -\varphi', -\mu')$—the diagonal image of v is clearly a pre-subkernel (see [2], Section 1.1). Now define $x_1' = x_1 \oplus - r_1 \oplus - r_2$ and $x_2' = x_2 \oplus - r_3$ (i.e., distinct r_i are contained in orthogonal summands of M). It is easy to verify that x_1' and x_2' map to the same two basis elements of $v \otimes \mathbf{Z}Q$ as x_1 and x_2, respectively, and $\varphi'(x_1', x_2') = \mu'(x_1') = \mu'(x_2') = 0$. We may now map the free module on the basis $\{x_i''\}$, $1 \leq i \leq 2k$, to M via a map sending x_i'' to x_i', $i \leq 2$, and x_i'' to x_i, $i > 2$, and pull back the quadratic form of v' to obtain an element of $\Gamma_i^s(f)$ that is, by Lemma 1.2 of [2], equivalent to v' and therefore to v.

We may clearly repeat this procedure a finite number of times so as to obtain a form that is strongly equivalent to zero in the sense of Section 1.1 of [2].

§3. THE ODD-DIMENSIONAL CASE

In this case we already know that the map $\Gamma_i^s(f) \to L_i^s(Q)$ is *injective* (see [2], Section 1.2) and we must show that it is also surjective. We will first recall the formation-theoretic description of Γ-groups and Wall groups due to Ranicki in [4]. Let F be a kernel over $\mathbf{Z}Q$; let R_1, R_2 be subkernels, and suppose that R_1 is a cononical subkernel. Then the triple $(F; R_1, R_2)$ is called a *formation* over $\mathbf{Z}Q$. If R_2 is also a standard subkernel the formation is said to be *trivial*—note that R_2 may be equivalent to R_1 or its complement (two subkernels are equivalent if there is a simple change of basis preserving the quadratic form and carrying one into the other). Two formation are simply-isomorphic if their kernels are simply-isomorphic via an isomorphism preserving the pair of subkernels. The direct sum of formations is defined by $(F_1; R_1, R_2) \oplus (F_2; S_1, S_2) = (F_1 \oplus F_2;\ R_1 \oplus S_1,\ R_2 \oplus S_2)$, and $L_i^s(Q)$ can be regarded as the group of stable simple isomorphism classes of formations over $\mathbf{Z}Q$. $\Gamma_i^s(f)$ can then be regarded as the subgroup generated by formations $(F; R_1, R_2)$ such that $(F; R_1, R_2) = (F'; R_1', T) \otimes_{\mathbf{Z}G} \mathbf{Z}Q$, where F' is a kernel over $\mathbf{Z}G$ with canonical subkernel R_1' and T is only required to be a *pre-subkernel* (see [2], Section 1.1) over f—we can call $(F'; R_1, T)$ a *pre-formation.*

We must show that every formation over $\mathbf{Z}Q$ lifts, modulo trivial formations, to a pre-formation. Let $(F; R_1, R_2)$ be a formation over $\mathbf{Z}Q$. Since $f\colon \mathbf{Z}G \to \mathbf{Z}Q$ is surjective, it follows that $(F;\ R_1, R_2) = (F'; R_1', M) \otimes_{\mathbf{Z}G} \mathbf{Z}Q$, where F' is a kernel over $\mathbf{Z}G$, R_1' a standard subkernel, and M the span of a set of elements of F' mapping to a basis of R_2—we will call these generators of $M\{x_i\}$. Note that M is not necessarily a pre-subkernel since the quadratic form induced on it by F' doesn't necessarily vanish identically. We will give an inductive procedure similar to that used in the even-dimensional case to modify M to make it a pre-subkernel. Let x_1, x_2 be two generators of M mapping to basis elements of R_2. Then $\varphi(x_1, x_2) = r_1$, $\mu(x_1) = r_2$, $\mu(x_2) = r_3$, where the r_i are contained in K. Form the sum $(F'\colon R_1', M) \oplus V(r_1) \oplus V(r_2) \oplus V(r_3)$ (see Lemma 1.3 and the discussion following it), where the module M is replaced by $M \oplus B(r_1) \oplus B(r_2) \oplus B(r_3)$. Let j_i, k_i be the elements of $V(r_i)$ defined in Lemma 1.3 and let $x_1' = x_1 \oplus j_1 \oplus k_2 - j_2$, $x_2' = x_2 \oplus -k_1 \oplus k_3 \oplus -j_3$. It is not difficult to verify that $\varphi(x_1', x_2') = \mu(x_1') = \mu(x_2') = 0$ (using the mutual orthogonality of the $B(r_i)$). Furthermore, we have

$$\varphi(x_i', B(r_t)) = \varphi(x_i, B(r_t)) \pm \varphi(j_s, B(r_t)) \pm \varphi(k_n, B(r_t))$$

where there may be two summands with j's or k's paired with $B(r_t)$. All such pairings must be zero either by orthogonality, the fact that j_s is contained in the same subkernel as $B(r_s)$ in $V(r_s)$, or the fact that $\varphi(k_n, B(r_n)) = 0$, by Lemma 1.3. It follows that the induced quadratic form on $M' \subset M \oplus B(r_1) \oplus B(r_2) \oplus b(r_3)$, where M' is the span of x_1', x_2', x_i, $i > 2$, $b(r_t)$ must vanish everywhere, except perhaps, on the span of x_i, $i > 2$, and the B – pre-subkernels. We can clearly continue this process until we arrive at a pre-subkernel in the direct sum of our original kernel F with a finite number of copies of pre-formations of the type $V(r)$. Since these pre-formations map to the trivial formation over $\mathbf{Z}Q$, the conclusion of the theorem follows.

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