Journal of Pure and Applied Algebra 24 (1982) 73–77
North-Holland Publishing Company 

# GROUP COHOMOLOGY AND EQUIVARIANT MOORE SPACES

Justin R. SMITH*
*Department of Mathematics, State University of New York at Binghamton, NY 13901, USA*

Communicated by A. Heller
Received 19 September 1980
Revised 17 February 1981

## 1. Introduction

This paper will describe a relation between the homology and cohomology groups of a group $\pi$ and the Tor- and Ext-functors of modules over the integral group ring $\mathbb{Z}\pi$. This relation will be applied to reformulate the obstruction theory to the existence of equivariant Moore spaces developed in [3] in a manner that may facilitate computation of the obstructions. Although the main theorem of this paper is a consequence of well-known results, I have never seen it explicitly stated before. I am indebted to Henri Cartan and Alex Heller for several important improvements in the statement and proof of the main result.

All modules in this paper will be assumed to be left $\mathbb{Z}\pi$-modules unless they appear in functors that require $\mathbb{Z}\pi$ to act on the *right* in which case they will be regarded as *right* $\mathbb{Z}\pi$-modules via the involution of $\mathbb{Z}\pi$ that maps group elements to their inverses.

**Theorem 1.1.** *Let $M$ and $N$ be $\mathbb{Z}\pi$-modules. Then there exist natural homomorphisms*

(1) $\eta_*:\operatorname{Tor}_i^{\mathbb{Z}\pi}(M,N)\to H_i(\pi,M\otimes N)$;

(2) $\xi^*:H^i(\pi,\operatorname{Hom}(M,N))\to\operatorname{Ext}^i_{\mathbb{Z}\pi}(M,N)$ *for all i.*

*If* $\operatorname{Tor}_1^{\mathbb{Z}}(M,N)=0$, *then $\eta_*$ is an isomorphism and if* $\operatorname{Ext}^1_{\mathbb{Z}}(M,N)=0$, *then $\xi^*$ is an isomorphism. Furthermore, if y denotes Yoneda products the following diagram commutes:*

$$\begin{array}{ccc}
H^i(\pi,\operatorname{Hom}(N,T))\otimes H^j(\pi,\operatorname{Hom}(M,N)) & \xrightarrow{\xi^*\otimes\xi^*} & \operatorname{Ext}^i_{\mathbb{Z}\pi}(N,T)\otimes\operatorname{Ext}^j_{\mathbb{Z}\pi}(M,N) \\
\downarrow{\scriptstyle\cup} & & \\
H^{i+j}(\pi,\operatorname{Hom}(N,T)\otimes\operatorname{Hom}(M,N)) & & \downarrow{\scriptstyle y} \\
\downarrow{\scriptstyle c_*} & & \\
H^{i+j}(\pi,\operatorname{Hom}(M,T)) & \xrightarrow[\xi^*]{} & \operatorname{Ext}^{i+j}_{\mathbb{Z}\pi}(M,T)
\end{array}$$

* Partially supported by National Science Foundation Grant No. MCS-79-00296.

0022-4049/82/0000–0000/$02.75 

*where $\cup$ denotes the cup product and $\boldsymbol{c}_*$ is the map induced in cohomology by the composition homomorphism*

$$\boldsymbol{c}:\mathrm{Hom}(N,T)\otimes_{\mathbb{Z}}\mathrm{Hom}(M,N)\to\mathrm{Hom}(M,T).$$

**Remarks.** 1. The proof will be deferred to Section 2.

2. In (1) above the $\mathbb{Z}$-tensor product $M\otimes N$ is equipped with the diagonal $\pi$-action. In (2) the $\pi$-action on $\mathrm{Hom}(M,N)$ is defined by $(g\alpha)(c)=g(\alpha(g^{-1}c))$ where $g\in\pi$, $\alpha\in\mathrm{Hom}(M,N)$ and $c\in M$.

3. If $\mathrm{Ext}^1(M,N)$, $\mathrm{Ext}^1(N,T)$ and $\mathrm{Ext}^1(M,T)$ are all 0, the diagram above may be used to compute Yoneda products.

4. The maps $\eta_*$ and $\xi^*$ are, respectively, the maps $\cup j$ and $\cap j$ in the statement of Proposition 9.3 on p. 227 of [1].

**Corollary 1.2.** *If $M$ is a $\mathbb{Z}$-free $\mathbb{Z}\pi$-module its (projective) homological dimension is $\le$ the cohomological dimension of $\pi$.* □

**Corollary 1.3.** *The global homological dimension of the ring $\mathbb{Z}\pi$ is $\le 1+$ the cohomological dimension of $\pi$. If the cohomological dimension of $\pi$ is $n$ then equality occurs if there exists a $\mathbb{Z}\pi$-module $M$ such that $H^n(\pi,M)$ is a nonzero, non-divisible abelian group.*

**Remarks.** Although the condition on $\pi$ in the second statement seems rather technical it is not hard to describe a class of groups that satisfy it:

**Proposition 1.4.** *If $\pi$ is a finite extension of a polycyclic group and has cohomological dimension $n$, then $H^n(\pi,\mathbb{Z}\pi)$ is a nontrivial finitely generated abelian group, hence non-divisible.*

**Remark.** The Mayer–Vietoris sequence implies that any *free-product* of groups that are finite extensions of polycyclic groups will have top cohomology groups (with coefficients in the group-ring) that are non-divisible, although they may be infinitely generated.

**Proof of Proposition 1.4.** The statement that $H^n(\pi,\mathbb{Z}\pi)$ is a finitely generated abelian group follows from a repeated application of the Lyndon spectral sequence and the fact that $H^*(\pi,A)$ is a finitely generated abelian group whenever $A$ is a finitely generated abelian group (with, possibly nontrivial $\pi$-action) or $\mathbb{Z}\pi$ and $\pi$ is finite or $\mathbb{Z}$. The statement that $H^n(\pi,\mathbb{Z}\pi)\neq 0$ follows from the fact that $\pi$ is an $(\overline{\mathrm{FP}})$-group in the sense of [4] so the trivial $\mathbb{Z}\pi$-module $\mathbb{Z}$ has a finitely generated projective resolution, say

$$0\to P_n\to\cdots\to P_0\to\mathbb{Z}\to 0$$

and $H^n(\pi, P_n)$ must be nonzero or the resolution would be reducible. The conclusion follows from the additivity of the functor $H^n(\pi, *)$ for finitely generated modules. □

**Proof of Corollary 1.3.** To prove the first statement note that if $B$ is *any* left $\mathbb{Z}\pi$-module there exists a short exact sequence $0 \to K \to F \to B \to 0$, where $F$ is free and $K$ is $\mathbb{Z}$-free. The conclusion follows from the fact that the homological dimension of $K$ is one less than that of $B$ and from Corollary 1.2 above.

To prove the second statement of Corollary 1.3 we need only exhibit a $\mathbb{Z}\pi$-module whose homological dimension is $n+1$. Let $A = H^n(\pi, M)$ be as in the second statement of Corollary 1.3 and let $p$ be an integer such that $A/pA \neq 0$ – such an integer exists by hypothesis.

*Claim. The trivial $\mathbb{Z}\pi$-module $\mathbb{Z}/p\mathbb{Z}$ has homological dimension $n+1$.*

This follows from the exact sequence

$$\cdots \to \operatorname{Ext}^i_{\mathbb{Z}\pi}(\mathbb{Z}, M) \xrightarrow{\cdot p} \operatorname{Ext}^i_{\mathbb{Z}\pi}(\mathbb{Z}, M) \to \operatorname{Ext}^{i+1}_{\mathbb{Z}\pi}(\mathbb{Z}/p\mathbb{Z}, M) \to 0$$

induced by $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$. The map induced in Ext by multiplication by $p$ is also multiplication by $p$ because multiplication of an entire projective resolution of $\mathbb{Z}$ by $p$ is a chain-map. This proves the corollary. □

We will now apply these results to the problem of the existence of equivariant Moore spaces. If $M$ is a $\mathbb{Z}\pi$-module and $n$ is an integer greater than 1, a connected CW-complex, $X$, is called a Moore space of type $(M, n; \pi)$ if:

(1) its fundamental group is $\pi$;

(2) $H_i(X; \mathbb{Z}\pi) = 0$ for $i \neq 0, n$;

(3) $H_0(X; \mathbb{Z}\pi) = \mathbb{Z}$;

(4) $H_n(X; \mathbb{Z}\pi) = M$.

Steenrod raised the question: For which triples $(M, n; \pi)$ do such spaces exist?

In [3] an obstruction theory was developed for the existence of Moore spaces. The obstructions to the existence of a Moore space of type $(M, n; \pi)$ are elements $c_i \in \operatorname{Ext}^{i+1}_{\mathbb{Z}\pi}(M, *)$, $i \geq 2$ – in fact the first nontrivial obstruction lies in the group $\operatorname{Ext}^3_{\mathbb{Z}\pi}(M, M/2M)$ if $n > 2$. Corollaries 1.2 and 1.3 imply:

**Corollary 1.5.** *If $\pi$ is a group of cohomological dimension $\leq 2$ and $M$ is a $\mathbb{Z}$-torsion free $\mathbb{Z}\pi$-module then there exist Moore spaces of type $(M, n; \pi)$ if $n > 1$ and if $M$ is $\mathbb{Z}$-torsion with $M/2M = 0$ there exist Moore spaces of type $(M, n; \pi)$ if $n > 2$.* □

**Remarks.** 1. The corollaries cited above imply that the *first* obstruction is the *only* obstruction.

2. In [6] Carlsson has given an example of a group $\pi$ and a $\mathbb{Z}\pi$-module $M$ such that *no* corresponding Moore spaces exist.

3. It is well known that all low-dimensional knot groups have cohomological dimension $\leq 2$ – see [5].

4. I understand that Peter Kahn has proved a similar result using different methods.

## 2. Proof of Theorem 1.1

The following is an extensive revision of my original argument incorporating suggestions of Alex Heller.

Let $\mathscr{A}$ be an abelian category and let $\mathscr{C}$ be the full subcategory of the category DG$\mathscr{A}$ of chain complexes in $\mathscr{A}$ containing those complexes whose homology is nonzero in finitely many degrees. $\mathscr{C}$ is a graded additive category whose underlying additive category is denoted $\mathscr{C}_0$. $\mathscr{A}$ imbeds as a full subcategory of $\mathscr{C}_0$ by mapping an object to the corresponding complex concentrated in degree 0. Let the quotient of $\mathscr{C}$ by the homotopy congruence be denoted by $\bar{\mathscr{C}}$; $\mathscr{A} \to \bar{\mathscr{C}}_0$ is, then, still a full imbedding. The homology functor $H: \mathscr{C} \to \mathrm{G}\mathscr{A}$ factors through $\bar{\mathscr{C}}$. A morphism $f$ in $\mathscr{C}$ or $\bar{\mathscr{C}}$ is a *weak equivalence* if $Hf$ is an isomorphism.

An object $X$ in $\mathscr{C}$ is called *P-cofibrant* if each $X_q$ is projective and, for some $n$, $X_q = 0$, $q < n$; the dual notion is *I-fibrant*. A projective resolution of an object $Y$ in $\mathscr{C}$ is a weak equivalence $X \to Y$ with $X$ P-cofibrant; injective resolutions are defined dually. If $\mathscr{A}$ has enough projectives (injectives) then any $Y$ has a projective (injective) resolution. It follows that $\bar{\mathscr{C}}$ admits a calculus of right (left) fractions with respect to the weak equivalences and thus a category of fractions $\mathscr{D} = \bar{\mathscr{C}}[\mathrm{w.e.}^{-1}] = \mathscr{C}[\mathrm{w.e.}^{-1}]$, with $\bar{\mathscr{C}}(X, Y) \to \mathscr{D}(X, Y)$ bijective if $X$ is P-cofibrant or $Y$ is I-fibrant.

$\mathscr{D}$ is essentially Verdier's 'derived category'. Inspection shows that $\mathscr{D}(M,N) = \mathrm{Ext}(M,N)$ for $M$ and $N$ in $\mathscr{A}$ and that the *Yoneda product* is just *composition* in $\mathscr{D}$.

Throughout the remainder of this section $\mathscr{A}$ will be the category of left $\mathbb{Z}\pi$-modules where $\pi$ is a group as in the statement of Theorem 1.1. Note that $\mathscr{A}$ is symmetric monoidal closed with respect to the tensor product $M, N \to M \otimes N$ over $\mathbb{Z}$, provided with the diagonal action, and the 'internal hom' $M, N \to \mathrm{Hom}(M,N)$ over $\mathbb{Z}$, with the 'conjugation' action – the unit is $\mathbb{Z}$ with the trivial action. Both of these functors extend in the usual way to DG$\mathscr{A}$.

The category $\mathscr{D}$ is also symmetric monoidal closed, with tensor product $X, Y \to X \underline{\otimes} Y$ and 'internal hom' $X, Y \to \mathbf{Hom}(X, Y)$ characterized by:

(1) $X \underline{\otimes} Y = X \otimes Y$ *if* $X$ *or* $Y$ *is P-cofibrant*;

(2) $\mathbf{Hom}(X, Y) = \mathrm{Hom}(X, Y)$ *if* $X$ *is P-cofibrant or* $Y$ *is I-fibrant*.

The unit is still $\mathbb{Z}$. The symmetric monoidal character provides a canonical morphism $\mathbf{Hom}(Y, W) \underline{\otimes} \mathbf{Hom}(X, Y) \to \mathbf{Hom}(X, W)$, called *composition* and given, in fact, by composition if $X$ is P-cofibrant and $W$ is I-fibrant. Furthermore there are canonical isomorphisms

$$\mathscr{D}(\mathbb{Z}, \mathbf{Hom}(X, Y)) \simeq \mathscr{D}(\mathbb{Z} \underline{\otimes} X, Y) \simeq \mathscr{D}(X, Y)$$

such that if $f: \mathbb{Z} \to \mathbf{Hom}(X, Y)$, $g: \mathbb{Z} \to \mathbf{Hom}(Y, W)$, then the composition in $\mathscr{D}$

$$\mathbb{Z} \simeq \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{g \otimes f} \mathbf{Hom}(Y, W) \otimes \mathbf{Hom}(X, Y) \to \mathbf{Hom}(X, W)$$

is the Yoneda product $g \curlyvee f$.

The cohomology of $\pi$ with coefficients in $M$ is, by definition, $H^*(\pi, M) = \mathscr{D}(\mathbb{Z}, M)$. If $\mu : M' \otimes M'' \to M$ in $\mathscr{A}$ the cup-product relative to $\mu$ of $f : \mathbb{Z} \to M'$, $g : \mathbb{Z} \to M''$ is the composition in $\mathscr{D}$

$$\mathbb{Z} \simeq \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{f \otimes g} M' \otimes M'' \xrightarrow{\mu} M.$$

If $M$ and $N$ are in $\mathscr{A}$ there is a canonical morphism

$$\xi : \mathrm{Hom}(M, N) \to \mathbf{Hom}(M, N) \text{ natural in } \mathscr{A} \text{ with } H_0\xi \text{ the identity.}$$

Composition with this induces

$$\xi^* : H^*(\pi, \mathrm{Hom}(M, N)) = \mathscr{D}(\mathbb{Z}, \mathrm{Hom}(M, N)) \to \mathscr{D}(\mathbb{Z}, \mathbf{Hom}(M, N)) \simeq \mathrm{Ext}(M, N)$$

and this defines the corresponding map in the statement of Theorem 1.1. For $M, N, T$ in $\mathscr{A}$ the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}(N, T) \otimes \mathrm{Hom}(M, N) & \longrightarrow & \mathrm{Hom}(M, T) \\ \downarrow \xi \otimes \xi & & \downarrow \xi \\ \mathbf{Hom}(N, T) \otimes \mathbf{Hom}(M, N) & \longrightarrow & \mathbf{Hom}(M, T) \end{array}$$

and the discussion above relating cup- and Yoneda-products with composition in $\mathscr{D}$ implies the commutativity of the diagram in Theorem 1.1.

Since $\mathbf{Hom}(M, N)$ (with $M$ and $N$ in $\mathscr{A}$) has homology $\mathrm{Hom}(M, N)$ in degree 0 and $\mathrm{Ext}_{\mathbb{Z}}(M, N)$ in degree $-1$, it follows that $\xi$ is an *isomorphism* (in $\mathscr{D}$) if and only if $\mathrm{Ext}_{\mathbb{Z}}(M, N) = 0$. Dually there is a canonical morphism $M \underline{\otimes} N \to M \otimes N$, giving rise to $\mathrm{Tor}(M, N) \to H_*(\pi, M \otimes N)$ with isomorphism when $\mathrm{Tor}^{\mathbb{Z}}(M, N) = 0$.

## References

[1] H. Cartan and S. Eilenberg, Homological Algebra (Princeton Univesity Press, Princeton, 1956).
[2] S. MacLane, Homology (Springer, Berlin–New York, 1975).
[3] J. Smith, Topological realization of chain complexes, Preprint.
[4] R. Bieri and B. Eckmann, Amalgamated free products of groups and homological duality, Comment. Math. Helv. 49 (1974) 460–478.
[5] L. Neuwith, Knot Groups (Princeton University Press, Princeton, 1965).
[6] G. Carlsson, A counterexample to a conjecture of Steenrod, Preprint.