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EQUIVARIANT MOORE SPACES II. THE LOW-DIMENSIONAL CASE

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Introduction

This paper forms a continuation of [9].

Let π be a group and let M be a right $\mathbb{Z}\pi$ module. A space of type $(M, 2; \pi)$ is a topological space X that satisfies the following conditions:

(1) $\pi_1(X) = \pi$,

(2) $H_0(X; \mathbb{Z}\pi) = \mathbb{Z}$,

(3) $H_i(X; \mathbb{Z}\pi) = 0, \quad i \neq 2$,

(4) $H_2(X; \mathbb{Z}\pi) = M$.

This paper considers the question of when such spaces exist, given π and M. This is a special case of the Steenrod problem that was dealt with in [9], which developed an obstruction theory for the existence of such spaces. In the special case studied here it turns out that additional obstructions exist to the formation of equivariant Moore spaces that are derived essentially from that fact that a topological space is a limit of multiplicative constructions (stages of a Postnikov tower). This additional obstruction is called the *multiplicative component* of the obstruction to the existence of equivariant Moore spaces. Multiplicative components exist in the general case too but only in the higher-order obstructions – they only enter into the *first obstruction* in the two dimensional case.

The multiplicative component of the obstruction is shown to be non-zero in its own right, and in the two-dimensional case it is shown that this obstruction can be *cancelled* by the introduction of a suitable first k-invariant in the topological spaces, in certain cases. As in the general case, the non-realizability results in the present paper can be phrased in terms of spaces that are not equivariant Moore spaces. A

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fairly typical result (whose statement doesn't require the use of any of the technical terms defined in Section 1) is the following:

Proposition. *Let X be a topological space with the following properties:*

(1) $\pi_1(X)=\pi$, *a group with the property that* $H^3(\pi,\mathbb{Z})\neq 0$ *and has no 2-torsion elements,*

(2) $H_0(X;\mathbb{Z}\pi)=\mathbb{Z}$,

(3) $H_2(X;\mathbb{Z}\pi)=\mathbb{Z}$,

(4) $H_3(X;\mathbb{Z}\pi)=H_4(X;\mathbb{Z}\pi)=0$.

Then the first k-invariant of X (an element of $H^3(\pi,\mathbb{Z})$*) must be zero.* □

Remarks. (1) This is 1.12 in the present paper.

(2) In this case the corresponding equivariant Moore space of type $(\mathbb{Z};2;\pi)$ also fits the conditions. The obstruction in [9] *vanishes* in this case – the significant obstruction is the multiplicative one.

(3) The proposition above can be rephrased as:

Proposition. *Let* π *be a group such that* $H^3(\pi;\mathbb{Z})\neq 0$ *and has no 2-torsion elements. Let* C_* *be a projective* $\mathbb{Z}\pi$*-chain complex with the following properties:*

(1) $H_0(C_*)=\mathbb{Z}$,

(2) $H_1(C_*)=0$,

(3) $H_2(C_*)=\mathbb{Z}$,

(4) $H_3(C_*)=H_4(C_*)=0$,

(5) *the first homological k-invariant of* C_* *is non-zero.*

Then C_* *is not chain-homotopy equivalent to the chain-complex of any topological space.* □

Section 1 of the present paper proves these results (and others whose statements are more technical) and develops the theory of the multiplicative component of the obstruction to realizing an equivariant Moore space.

Section 2 proves a technical result that may be of some independent interest. It constructs an equivariant left-inverse to the map constructed from their bar construction to their W-construction, and shows that the reverse composite of the two maps is equivariantly homotopic to the identity so the bar and W-constructions are equivariantly homotopy equivalent. This implies the corresponding statement about the Eilenberg–MacLane model for Eilenberg–MacLane spaces and the model due to Milgram.

1. The multiplicative obstruction

Definition 1.1. Let M be a $\mathbb{Z}\pi$-module an let $K(M,2)$ be the Eilenberg–MacLane space on M equipped with the appropriate π-action. Then $\mathscr{G}_M\in \mathrm{Ext}^3_{\mathbb{Z}\pi}(M,\Gamma(M))$ is defined to be the first homological k-invariant of the chain-complex $K(M,2)_*\otimes Z_*$, where Z_* is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$. □

Remarks. (1) Something related to the class $\mathscr{G}_M$ was defined by K. Igusa in [6] in the case where M was $\mathbb{Z}$-torsion free and π was the general linear group over $\mathbb{Z}$ of a suitable degree. He showed that in the case where $M=\mathbb{Z}^3$ and π is $\mathrm{GL}_3(\mathbb{Z})$ acting in the usual way, that $\mathscr{G}_M\neq 0$.

Igusa calls his construction the Grassman invariant of M and this term will be used throughout the present paper for $\mathscr{G}_M$.

(2) An algorithm is given for computing the Grassman invariant of M in the case where M is $\mathbb{Z}$-torsion free and an explicit formula is given in the case where $M=\mathbb{Z}^3$. The invariant is computed in the case where $M=\mathbb{Z}^3$, $\pi=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ with generators acting via right multiplication by the respective matrices

$$\begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & -1\\ 0 & 0 & -1\end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & 0\\ -1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}.$$

The Grassman invariant of M is nonzero – in fact its image in $\mathrm{Ext}^3(M,M/2M)$ under the change of coefficients homomorphism induced by the canonical map $\Gamma(M)\to M/2M$ is also nonzero.

(3) The work of Peter Kahn implies that $\mathscr{G}_M=0$ whenever the underlying abelian group of M is $\mathbb{Z}\oplus\mathbb{Z}$.

Recall that $\Gamma(M)$ can be regarded as the free abelian group generated by symbols $\{\gamma(m), m\in M\}$ subject to the quadratic identity:

$$\gamma(m_1+m_2+m_3)-\gamma(m_1+m_2)-\gamma(m_1+m_3)-\gamma(m_2+m_3)$$
$$+\gamma(m_1)+\gamma(m_2)+\gamma(m_3)=0$$

for all $m_1,m_2,m_3\in M$. Let $L_M: M\otimes M\to\Gamma(M)$ be the map that sends $m_1\otimes m_2$ to $\gamma(m_1+m_2)-\gamma(m_1)-\gamma(m_2)$ – this map can be shown to be bilinear.

Let M be a $\mathbb{Z}\pi$-module and let P_* be a projective resolution of M. If Z_* is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ the Künneth theorem implies that $Z_*\otimes P_*$ is also a projective resolution of M so that there exists a unique chain-homotopy class of chain maps $h: P_*\to Z_*\otimes P_*$ inducing the identity map of M. Let $\varepsilon: P_*\to M$ be the augmentation.

Definition 1.2. Under the hypotheses above, let $x\in H^i(\pi,M)$ be a cohomology class with $i>1$. Define $\mu_M(x)=\mathrm{Ext}^i_{\mathbb{Z}\pi}(M,\Gamma(M))$ to be the class defined by $L_M\circ(x\otimes\varepsilon)\circ h$. □

Remarks. (1) This is something like an 'external cup product' with the identity map of M.

(2) In [3] the following sequence is shown to be exact

$$0 \to \Lambda(M) \to M \otimes M_{L_M} \to \Gamma(M) \to M/2M \to 0;$$

where $\Lambda(M)$ is the submodule generated by elements $\{m_1 \otimes m_2 - m_2 \otimes m_1, m_1, m_2 \in M\}$.

Definition 1.3. Let M be a $\mathbb{Z}\pi$-module and let $x \in H^3(\pi, M)$ be a cohomology class, represented by a cocycle $\bar{x}: Z_3 \to M$, where Z_* is a projective $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$. Define $\Delta(x)$ to be the desuspension of the algebraic mapping cone of the chain map

$$Z_* \to \Sigma^3 P_*$$

that extends $\bar{x}$, where p_* is a projective resolution of M. □

Remarks. (1) It is not hard to see that the chain-homotopy type of $\Delta(x)$ doesn't depend upon any of the choices that have been made.

(2) It is also clear that the homology of $\Delta(x)$ is as follows:

$$H_0(\Delta(x)) = \mathbb{Z}, \qquad H_2(\Delta(x)) = M, \qquad H_i(\Delta(x)) = 0, \quad i \neq 0, 2,$$

and that the first homological k-invariant is precisely x.

The theory of homological k-invariants in [5] implies that if C_* is a chain-complex with $H_1(C_*) = 0$, and isomorphisms $\alpha: H(C_*) \to \mathbb{Z}$, $\beta: H_2(C_*) \to M$, then there exists a chain map $h: C_* \to \Delta(x)$ inducing α and β in the appropriate dimensions if and only if the first homological k-invariant of $C_* \in \operatorname{Ext}^3_{\mathbb{Z}\pi}(H_0(C_*), H_2(C_*))$ is precisely $(\alpha^*)^{-1}x(\beta^*)$. If this map h exists it is unique up to a chain-homotopy.

(3) If $H_i(C_*) = 0$, $2 < i < n$, then the second non-trivial homological k-invariant of C_* lies in $H^{n+1}(\Delta(x); H_n(C_*))$ and is the pullback of the class $1: H_n(C_*) \to H_n(C_*)$ in $H^{n+1}(\mathscr{A}(h); H_n(C_*)) = \operatorname{Hom}_{\mathbb{Z}\pi}(H_n(C_*), H_n(C_*))$ over the standard inclusion $i: \Delta(x) \to \mathscr{A}(h)$. Here $\mathscr{A}(h)$ is the algebraic mapping cone of $h: C_* \to \Delta(x)$ and the statement about $H^{n+1}(\mathscr{A}(h); H_n(C_*))$ follows from the fact that $H_i(\mathscr{A}(h)) = 0$, $i < n+1$. If C_* is the chain complex of a $K(M, 2)$-fibration over a $K(\pi, 1)$ with characteristic class x, then $n = 4$ and the second homological k-invariant lies in $H^5(\Delta(x); \Gamma(M))$ (where $\Gamma(M) = H_4(K(M, 2))$ – see [3]). This homological k-invariant was shown in [9] to be precisely the obstruction to killing $H_4(C_*)$ by taking a fibration over its space or the first obstruction to forming an equivariant Moore space of type $(M, 2; \pi)$.

Definition 1.4. Let C_i, $i = 1, 2$ be chain complexes. A *contraction of* C_1 *onto* C_2 is a triple (p, q, Ξ) where $p: C_1 \to C_2$ and $q: C_2 \to C_1$ are chain-maps such that $p \cdot q = 1: C_1 \to C_1$ and Ξ is a chain-homotopy from the identity map of C_2 to $q \cdot p$. These maps are required to satisfy the additional condition that $\Xi \circ q = 0$, $p \circ \Xi = 0$ and $\Xi^2 = 0$. □

Remark. The condition that $\Xi^2=0$ didn't appear in the original definition of Eilenberg and MacLane in [2] but is necessary for the applications in the present paper.

Lemma 1.5 (Perturbation Lemma). *Let* $(f,g,\Phi):(C_1,d_1)\to(C_2,d_2)$ *be a contraction of chain complexes and let* d_1' *be a second differential on* C_1 *with* $t=d_1'-d_1$. *Suppose there exists a fibration on* C_1 *bounded from below and such that*
(a) t lowers fibration degree;
(b) Φ *and* d_1 *preserve it.*
Then there exists a second differential d_2' *on* C_2 *and a contraction*

$$(f',g',\Phi'):(C_1,d_1')\to(C_2,d_2')$$

where

$$T_\infty=1+\sum_{i=1}^{\infty}(\Phi t)^i, \tag{1}$$

$$f'=f\circ(1+t\circ T_\infty\circ\Phi), \tag{2}$$

$$g'=T_\infty\circ g, \tag{3}$$

$$d_2'=d_2+f\circ t\circ T_\infty\circ g, \tag{4}$$

$$\Phi'=T_\infty\circ\Phi. \quad \square \tag{5}$$

Remarks. (1) Note that, on account of the filtration on C_1, all of the 'infinite series' above reduce to a finite number of terms when evaluated on any element of C_1.

(2) This lemma first appeared in [4] though it was used implicitly in [8].

We will recall the concept of an F-extension of a map from [9]:

Definition 1.6. Let M and N be $\mathbb{Z}\pi$-modules, let F be a free $\mathbb{Z}\pi$-module with preferred basis $\{y_i\}$ and let $f:M\to N$ be a homomorphism of abelian groups that doesn't necessarily preserve the action of π. Then the *F-extension* of f, denoted $\tilde{f}_F:M\otimes_{\mathbb{Z}}F\to N\otimes_{\mathbb{Z}}F$ (equipped with the diagonal π-action) is defined by

$$\tilde{f}_F(m\otimes(y_i\cdot \upsilon))=f(m\cdot \upsilon^{-1})\cdot \upsilon\otimes(y_i\cdot \upsilon)$$

for all $m\in M$ and $\upsilon\in\pi$. $\square$

Remarks. (1) It is clearly possible to extend $\tilde{f}_F$ to all of $M\otimes F$, $\mathbb{Z}$-linearly. The resulting map is a $\mathbb{Z}\pi$-module homomorphisms.

(2) The F-extension of f defined above clearly depends upon the basis for F used. If f is *already* a module homomorphism, then $\tilde{f}_F=f\otimes 1$.

(3) The above definition clearly extends to *chain-complexes*. In this case bases for the chain modules of F must be defined in each dimension. If f is initially a chain-map its F-extension will also be a chain-map if the differential on F is 0.

Let $\xi \in H^3(\pi, M)$ be a cohomology class, where M is a $\mathbb{Z}\pi$-module and let X be the total space of the fibration with base a $K(\pi,1)$, fiber a $K(M,2)$ and with characteristic class ξ. As a semi-simplicial complex this is just a twisted cartesian product $K(\pi,1)\times_{\hat{\xi}} K(M,2)$ where $\hat{\xi}$ is the composite of the map of spaces $K(\pi,1)\to K(M,3)$ induced by ξ (this can be explicitly constructed semi-simplically) and the canonical annihilating twisting-function $K(M,3)\to K(M,2)$ – see [1, exposés 12 and 18].

The results of [4] imply that the *chain complex* of X is chain-homotopy equivalent to a suitable twisted tensor product $K(\pi,1)\otimes_{\bar{\xi}} K(M,2)$, where $\bar{\xi}$ agrees with ξ in dimension 3. We will now carry out a procedure similar to that used in the proofs of 2.5 and 2.6 in [9].

At this point we will make the simplifying assumption that the module M is $\mathbb{Z}$-torsion free.

Definition 1.7. If M is a $\mathbb{Z}\pi$-module $U(M)$ is a DGA-algebra concentrated in even degrees and generated by symbols $\gamma_i(x)$, in dimension $2i$ where i is an integer ≥ 0 and $x \in M$. These symbols are subject to the relations:

(1) $\quad \gamma_0(x)=1$ for all $x\in M$,

(2) $\quad \gamma_i(kx)=k^i\gamma_i(x)$, $k\in\mathbb{Z}$ for all i and $x\in M$,

(3) $\quad \gamma_i(x+y)=\sum_{j=0}^{i}\gamma_j(x)\gamma_{i-j}(y)$ for all $x,y\in M$ and all i,

(4) $\quad \gamma_i(x)\gamma_j(x)=\binom{i+j}{i}\gamma_{i+j}(x)$ for all i,j and all $s\in M$. □

Remarks. (1) This definition first appeared in [3] except that $U(M)$ was denoted $\Gamma(M)$. Our terminology follows that of Cartan in [1].

(2) It is not difficult to see that $U(M)_2$, generated by symbols $\gamma_1(x), x\in M$, may be identified with M itself and $U(M)_4$ can be identified with the Whitehead functor $\Gamma(M)$. It is also not difficult to see that the π-action on M extends in a natural way to $U(M)$.

(3) Note that the product of two elements $x,y\in M$, as defined in $U(M)$ is $\gamma_2(x+y)-\gamma_2(x)-\gamma_2(y)$, due to relation (3) in the definition.

(4) Due to our assumption that M is $\mathbb{Z}$-torsion free the DGA-algebra $U(M)$ is precisely the homology algebra of $K(M,2)$ – see Section 21 of [3].

In fact there exists a *contraction*

$$(p,q,\Psi): A(M,2)\to U(M).$$

This is a composite of several contractions:

(1) The contraction in the Direct Product Theorem in Section 6 of [3] (here we regard M as a direct sum of copies of $\mathbb{Z}$).

(2) A tensor product of contractions from $A(\mathbb{Z}, 1)$ to $A(x)$ defined in Section 14 of [3], after the bar construction has been taken.

Note that since the decomposition of M into a direct sum of copies of $\mathbb{Z}$ depends upon a choice of basis, it isn't clear that the maps in (p, q, Ψ) are module homomorphisms; and in most cases they aren't.

The maps p and q are, respectively, the maps $\mathfrak{a}$ and $\mathfrak{b}$ in [9]. In that paper the maps were explicitely computed in low dimensions.

We will perform a procedure similar to that used to prove 2.6 in [9] to compute the second homological k-invariant of $K(\pi,1)\otimes_{\xi} K(M,2)$ – see Remark (2) following 1.3.

Let Z_* denote the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$. Since M is $\mathbb{Z}$-free it follows that $M\otimes Z_*$, equipped with the diagonal π-action is a free $\mathbb{Z}\pi$-resolution of M. In the definition of $\Delta(\xi)$ assume the free resolution P_* of M used is $M\otimes Z_*$ (see 1.3).

Our main result is:

Theorem 1.8. *Under the hypotheses above the second homological k-invariant of $K(\pi,1)\otimes_{\xi} K(M,2)$ (and, here, that of the space X) is an element $x\in H^5(\Delta(\xi); \Gamma(M))$ represented as follows:*

(1) $$x\,|\,M\otimes Z_3=\mathscr{G}_M+\mu_M(\xi): M\otimes Z_3\to\Gamma(M);$$

(2) $$\begin{aligned} x\,|\,Z_5 &= \tilde{p}_Z\circ(\mathfrak{f}_2\otimes 1)\circ\xi_5\circ(\mathfrak{g}_2\otimes 1)\circ\tilde{q}_Z\\ &\quad+\tilde{p}_Z\circ\{(1\otimes d_Z)\circ\tilde{\Psi}_Z\}^2\circ(\mathfrak{f}_2\otimes 1)\circ\eta_3\circ(\mathfrak{g}_2\otimes 1)\circ\tilde{q}_Z\\ &: Z_5\to K_4\otimes Z_0\xrightarrow{1\otimes\varepsilon}\Gamma(M).\end{aligned}$$

Here $(\mathfrak{f}_2, \mathfrak{g}_2, \Phi_2): K(M,2)\to A(M,2)$ is the $\mathbb{Z}\pi$-contraction developed in Section 2 of the present paper, $\varepsilon: Z_0\to\mathbb{Z}$ is the augmentation, and η_3 is the composite.

$$K(M,2)\otimes Z_5=Z_5\xrightarrow{\Delta}(Z_*\otimes Z_*)_5\xrightarrow{\mathfrak{p}}Z_3\otimes Z_2\xrightarrow{\xi\otimes 1}K(M,2)_2\otimes Z_2$$

where Δ is the coproduct on Z_* and $\mathfrak{p}$ is the projection onto the direct summand.

Remarks. (1) Strictly speaking $\mathscr{G}_M$ and $\mu_M(\xi)$ are classes in $\mathrm{Ext}^3_{\mathbb{Z}\pi}(M,\Gamma(M))$.

In the statement of this theorem we are using those terms to denote the following cochains:

(A) $\mu_M(\xi)$ as defined in 1.2.

(B) $\mathscr{G}_M=\tilde{p}_Z\circ(1\otimes d_Z)\circ\{\tilde{\Psi}_Z\circ(1\otimes d_2\}^2\circ\tilde{q}_Z$ where $\tilde{p}_Z, \tilde{q}_Z$ are Z_*-extensions of maps, as defined in 1.6 and d_Z is the boundary homomorphism of Z_* – this formula for the first k-invariant of $K(M,2)\otimes Z_*$ was derived in [9].

(2) Consider the natural inclusion $\Sigma^2 M\otimes Z_*\to\Delta(\xi)$. This induces a homomorphism $H^5(\Delta(\xi); \Gamma(M))\to\mathrm{Ext}^3_{\mathbb{Z}\pi}(M,\Gamma(M))$ and statement (1) implies that the image of x under this map is precisely $\mathscr{G}_M+\mu_M(\xi)$. This is different from the results in the higher-dimensional case in [9] and shows that it is possible (in principal, at least)

for the homological k-invariant of $K(M,2)_*\otimes Z_*$ to be *cancelled* by the first k-invariant of the topological space X. The term $\mu_M(\xi)$ will be called the *multiplicative component* of the obstruction to killing the 4-dimensional homology of X.

Proof. Consider the contraction

$$(p\circ \mathfrak{f}_2\otimes 1, \mathfrak{g}_2\circ q\otimes 1, \Phi\otimes 1+\mathfrak{g}_2\circ\Psi\circ\mathfrak{f}_2\otimes 1):K(M,2)\otimes K(\pi,1)\to U(M)\otimes Z_*$$

where $K(\pi,1)_*=Z_*$. The maps in this contraction don't preserve the action of π, except for $\mathfrak{f}_2$, $\mathfrak{g}_2$, and Φ. If we take Z_* extensions of all maps (see 1.6) we get maps that preserve the action of π but are no longer chain maps, unless the boundary maps of Z_* vanish in all dimensions. Thus we get a $\mathbb{Z}\pi$-contraction

$$(\tilde p_Z\circ\mathfrak{f}_2, \mathfrak{g}_2\circ\tilde q_Z, \Phi+\mathfrak{g}_2\circ\tilde\Psi_Z\circ\mathfrak{f}_2)\otimes 1:K(M,2)\otimes(Z_*,0)\to U(M)\otimes(Z_*,0).$$

where $(Z_*,0)$ denotes a chain-complex whose chain-modules are the same as Z_* but whose boundary homomorphisms are *zero* and $\tilde v_Z\otimes 1$ denotes $\tilde v_Z$ where $v=p$, q, or Ψ.

Now we apply the Perturbation Lemma to this with the purturbation $t=1\otimes d_Z+\eta$, where η is the twisted portion of the boundary of the twisted tensor product $K(M,2)\otimes_\xi Z_*$ (which we have written as fiber x base rather than base x fiber) – this is equal to the composite $(m\otimes 1)\circ(1\otimes\xi\otimes 1)\circ(1\otimes c)$, with $m:K(M,2)_*\times K(M,2)_*\to K(M,2)_*$ the multiplication and $c:Z_*\to Z_*\otimes Z_*$ is the coproduct.

The result is a contraction

$$(f,g,\Theta):K(M,2)\otimes_\xi Z_*\to(U(M)\otimes Z_*,d')$$

with

(1) $$d'=\tilde p_Z\circ(\mathfrak{f}_2\otimes 1)\circ(1\otimes d_Z)\circ\left(1+\sum_{j=1}^{\infty}T^j\right)\circ(\mathfrak{g}_2\otimes 1)\circ\tilde q_Z,$$

$$T=\{\Phi\otimes 1+(\mathfrak{g}_2\otimes 1)\circ\tilde\Psi_Z(\mathfrak{f}_2\otimes 1)\}\circ\{1\otimes d_Z+\eta\},$$

(2) $$g=\left(1+\sum_{j=1}^{\infty}T^j\right)\circ(\mathfrak{g}_2\otimes 1)\circ\tilde q_Z,$$

(3) $$\theta=\left(1+\sum_{j=1}^{\infty}T^j\right)\circ(\Phi\otimes 1+(\mathfrak{g}_2\otimes 1)\circ\tilde\Psi_Z\circ(\mathfrak{f}_2\otimes 1)).$$

We will be mainly concerned with d'. Making use of the fact that $U(M)$ is concentrated in even degrees, and the fact that ξ vanishes below dimension 3 (so η *lowers* the dimension of the Z_* factor by at *least* 3) we find (from a tedious but straightforward computation) that $d'=1\otimes d_2$ *except* in the following cases:

1.9. (1) $\tilde p_Z\circ(\mathfrak{f}_2\otimes 1)\circ\eta_3\circ(\mathfrak{g}_2\otimes 1)\circ\tilde q_Z:U(M)_0\otimes Z_i\to U(M)_2\otimes Z_{i-3},\quad i\geq 3,$

(2) $\tilde p_Z\circ(\mathfrak{f}_2\otimes 1)\circ\eta_3\circ(\mathfrak{g}_2\otimes 1)\circ\tilde q_Z:U(M)_2\otimes Z_3\to U(M)_4\otimes Z_\infty,$

(3) $\mathscr{G}_M:U(M)_2\otimes Z_3\to U(M)_4\otimes Z_0,$

(4) $\quad \tilde{p}_Z \circ (\mathfrak{f}_2 \otimes 1) \circ \eta_5 \circ (\mathfrak{g}_2 \otimes 1) \circ \tilde{q}_Z : U(M)_0 \otimes Z_5 \to U(M)_4 \otimes Z_0,$

(5) $\quad \tilde{p}_Z \circ \{(1 \otimes d_Z) \circ \tilde{\Psi}_Z \circ (1 \otimes d_Z) \circ \tilde{\Psi}_Z\} \circ (\mathfrak{f}_2 \otimes 1) \circ \eta_3 \circ (\mathfrak{g}_2 \otimes 1) \circ \tilde{q}_Z$

$: U(M)_0 \otimes Z_5 \to U(M)_4 \otimes Z_0,$

where

(A) Terms (2) and (3) are to be added together;

(B) $\mathscr{G}_M = \tilde{p}_Z \circ (1 \otimes d_Z) \circ \{\tilde{\Psi}_Z \circ (1 \otimes d_Z)\}^2 \tilde{q}_Z$ (see [9]);

(C) η_3 is the term in η that lowers the dimension of the Z_* factor by 3 (i.e., in the formula for η only the term in the coproduct of Z_* that lowers the right factor by 3 is used);

(D) η_5 is the term in η that lowers the dimension of the Z_* factor by 5 (defined like η_3).

Since $U(M)_0 = \mathbb{Z}$, term 1 implies the existence of a copy of $\Delta(\Psi)$ (not necessarily a subcomplex) in $(U(M) \otimes Z_*, d')$, where $\Psi = \tilde{p}_Z \circ (\mathfrak{f}_2 \otimes 1) \circ \eta_3 \circ (\mathfrak{g}_2 \otimes 1) \circ \tilde{q}_Z$. Since q and $\mathfrak{g}_2$ are the *identity map* in dimension 0 and since $\tilde{p}_Z \circ (\mathfrak{f}_2 \otimes 1)$ sends $([M] \otimes 1 \otimes 1) \otimes Z$ in $K(M,2) \otimes Z_*$ to $\gamma_1(m) \otimes Z$ in $U(M)_2 \otimes Z_*$ it follows that Ψ can be identified with the original cocycle ξ, so $(U(M) \otimes Z_*, d')$ contains a copy of $\Delta(\xi)$.

In order to compute the 5-dimensional homological k-invariant of $D_* = (U(M) \otimes Z_*, d')$ (i.e., the *second nontrivial* k-invariant – see Remark (3) following 1.3) we map D_* to $\Delta(\xi)$ via the map that is the identity on $\Delta(\xi)$ and the zero map on $U(M)_4 \otimes Z_*$. The algebraic mapping cone, $\mathscr{A}$, of this map clearly has vanishing homology below dimension 5. It contains a subcomplex $\Sigma U(M)_4 \otimes Z_*$ whose inclusion induces an isomorphism in homology (at least in dimension ≤ 5). It follows that the cocycle $1_{H_5(\mathscr{A})} \in H^5(\mathscr{A}; H_5(\mathscr{A}))$ has the property that its *restriction* to $\Sigma U(M)_4 \otimes Z_*$ is precisely $\Sigma(1 \otimes \varepsilon)$ where ε is the augmentation of Z_*. In order for this map to be a cocycle though, its composite with the boundary from the next higher dimension must be 0. This boundary is

$$\begin{pmatrix} \Sigma d_\Delta & \psi & -1 \\ 0 & d_U & 0 \\ 0 & 0 & d_\Delta \end{pmatrix} : \{\Sigma(\Delta(\xi) \oplus U(M)_4 \otimes Z_*) \oplus \Delta(\xi)\}_6$$
$$\to \{\Sigma(\Delta(\xi) \oplus U(M)_4 \otimes Z_*) \oplus \Delta(\xi)\}_5$$

where d_Δ is the boundary of $\Delta(\xi)$, d_U that of $U(M)_4 \otimes Z_*$ and ψ represents the maps defined in terms (2+3), (4), (5) in 1.9 above. It is not difficult to see that we can define a cocycle on $\mathscr{A}$ if we define it to *vanish* on $\Sigma\Delta(\xi)$ and to equal ψ on $\Delta(\xi)$. It follows that the homological k-invariant in question is the cocycle on $\Delta(\xi)$ equal to $(1 \otimes \xi) \circ \psi$.

The proof of the theorem is almost complete: it only remains to be shown that term (2) in 1.9 is equal to $\mu_M(\xi)$, after being composed with $1 \otimes$the equation of Z_*.

First of all, note that in dimension 2 the map p *preserves* the action of π, i.e., if $m = \sum A_i x_i$, where the A_i are integers and the x_i are preferred $\mathbb{Z}$-basis elements of M, then p maps $[m]$ to $\sum A_i \gamma_i(x_i) \in U(M)_2$ (see 6.1 in [3], where p corresponds to f in that theorem). Since p also *preserves products* it follows that it preserves the action of π on the submodule $\mathscr{P}$ of $A(M,2)_4$ generated by products of 2-dimensional elements. Thus, $\tilde{p}_Z|_{\mathscr{P}} = p|_{\mathscr{P}}$.

Now suppose we apply term (2) in 1.9:

$$\tilde{p}_Z \circ (\mathfrak{f}_2 \otimes 1) \circ \eta_3 \circ (\mathfrak{g}_2 \otimes 1) \circ q_Z$$

to an element $u \otimes z$, where $u \in U(M)_2$ and z is a preferred basis element of Z_* (i.e., one used in the construction of Z_*-extensions of maps – see 1.6). Then $\tilde{q}_Z(u \otimes z) = q(u) \otimes z$ and $\mathfrak{g}_2 \otimes 1$ gives us $\mathfrak{g}_2 \circ q(u) \otimes z$. In dimension 2 $\mathfrak{g}_2$ is essentially the identity map, i.e., it sends $[m]$ to $[m] \otimes 1 \otimes 1 \in K(M,2)_2$. The term η_3 gives $(\mathfrak{g}_2 \circ q(u)) \circ \xi(z) \otimes \alpha$ where α is an element of π (this is due to the nature of the coproduct on the bar resolution Z_*).

Now $\mathfrak{f}_2$ preserves products in dimension 4 since it is an inverse to $\mathfrak{g}_2$, $\mathfrak{g}_2$ preserves products and since all of $K(M,2)_2$ is in the image of $\mathfrak{g}_2$. It follows that the application of $\mathfrak{f}_2$ gives

$$q(u) \circ \mathfrak{f}_2(\xi(Z)) \otimes \alpha \in A(M,2)_4 \otimes Z_0$$

(we have cancelled $\mathfrak{g}_2$). Now, because of the remark above about p preserving the action of π on products we get

$$u \circ (p \circ \mathfrak{f}_2 \circ \xi(Z)) \otimes \alpha \in U(M)_4 \otimes Z_0.$$

But, as was remarked earlier in the proof of this theorem, $p \circ \mathfrak{f}_2$ in dimension 2 can be regarded as the identity map since, if $m = \sum A_i x_i M$, it sends $[m] \otimes 1 \otimes 1 \in K(M,2)_2$ to $\sum A_i y_i(x_i) \in U(M)_2$. So we can identify $p \circ \mathfrak{f}_2 \circ \xi$ with ξ (by abuse of notation). After composing all this with $1 \otimes$the augmentation of Z_* we get $\mu_M(\xi)$ (since α gets mapped to 1). □

Corollary 1.10. *If X is a topological space with the following properties:*

(1) $\pi_1(X) = \pi$,

(2) $H_0(X; \mathbb{Z}\pi) = \mathbb{Z}$,

(3) $H_2(X; \mathbb{Z}\pi) = M$, *a $\mathbb{Z}$-free $\mathbb{Z}\pi$-module,*

(4) $H_3(X; \mathbb{Z}\pi) = H_4(X; \mathbb{Z}\pi) = 0$.

Then the first k-invariant of X, $\xi \in H^3(\pi, M)$ has the property that

$$\mu_M(\xi) = -\mathscr{G}_M \in \operatorname{Ext}^3_{\mathbb{Z}\pi}(M, \Gamma(M)). \qquad \square$$

Remarks. (1) This follows from the fact that the first two steps in constructing a Postnikov tower for X are the same as those required to construct an equivariant Moore space of type $(M, 2; \pi)$.

(2) It would seem that there is an indeterminacy in the definitions of ξ and $\mathscr{G}_M$ that should be taken into account. This indeterminacy is only apparent – in a

manner of speaking $\mathscr{G}_M$ is defined in terms of ξ. Pick a projective resolution Z_* of $\mathbb{Z}$ and a representative ξ of the first k-invariant – this is a cocycle

$$\xi : Z_3 \to M.$$

If we take the $\mathbb{Z}$-tensor product with M we get a representative of $\mu_M(\xi)$:

$$M \otimes Z_3 \xrightarrow{1\otimes\xi} M \otimes M \to \Gamma(M)$$

This setting can also be used to define $\mathscr{G}_M$, however:

$$M \otimes Z_3 \xrightarrow{\tilde{q}_Z} A(M,2)_2 \otimes Z_3 \xrightarrow{1\otimes d_Z} A(M,2)_2 \otimes Z_2$$
$$\xrightarrow{\tilde{\Psi}_Z} A(M,2)_3 \otimes Z_2 \xrightarrow{1\otimes d_Z} A(M,2)_3 \otimes Z_1 \xrightarrow{\tilde{\Psi}_Z} A(M,2)_4 \otimes Z,$$
$$\xrightarrow{1\otimes d_Z} A(M,2)_4 \otimes Z_0 \xrightarrow{\tilde{p}_Z} \Gamma(M) \otimes Z_0 \xrightarrow{1\otimes\varepsilon} \Gamma(M).$$

An automorphism of M or a change in Z_* by a chain homotopy equivalence will produce compensating changes in $\mathscr{G}_M$, if it is computed as indicated above.

(3) Since the image of $\mu_M(\xi)$ under the homomorphism induced by a change of coefficients

$$b : \mathrm{Ext}^3_{\mathbb{Z}\pi}(M, \Gamma(M)) \to \mathrm{Ext}^3_{\mathbb{Z}\pi}(M, M/2M)$$

is *zero* it follows that the only modules M that can occur in the setting of 1.10 are those modules for which $b(\mathscr{G}_M) = 0$. Since that *isn't* true for the module described in Remark (2) following 1.1 we get:

Corollary 1.11. *There doesn't exist a topological space X with the following properties:*

(1) $\pi_1(X) =$ *the group* $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ *with generators s and t,*

(2) $H_0(X; \mathbb{Z}\pi) = \mathbb{Z}$,

(3) $H_2(X; \mathbb{Z}\pi) =$ *the module with underlying abelian group* $\mathbb{Z}^3$ *and with s acting via right multiplication by*

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

and t acting via multiplication by

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

(4) $H_3(X; \mathbb{Z}\pi) = H_4(X; \mathbb{Z}\pi) = 0$. □

It isn't difficult to see that, if M is the module $\mathbb{Z}$ with *trivial* π-action then

$\mathscr{G}_M=0$. Coupled with the fact that $\mu_M(\xi):H^3(\pi;\mathbb{Z})\to\operatorname{Ext}^3_{\mathbb{Z}[\pi]}(\mathbb{Z},\mathbb{Z})=H^3(\pi;\mathbb{Z})$ is just multiplication by 2, we get

Corollary 1.12. *If X is a topological space with*
(1) $\pi_1(X)=\pi$, *a group with the property that* $H^3(\pi,\mathbb{Z})\neq 0$ *and*
(2) $H_0(X;\mathbb{Z}\pi)=\mathbb{Z}$,
(3) $H_2(X;\mathbb{Z}\pi)=\mathbb{Z}$,
(4) $H_3(X;\mathbb{Z}\pi)=H_4(X;\mathbb{Z}\pi)=0$,
then the first k-invariant of X is 0. □

Since the first k-invariant of a space is the same as the first homological k-invariant of its chain complex, this statement is equivalent to:

Corollary 1.13. *If π is a group such that $H^3(\pi,\mathbb{Z})\neq 0$ and has no 2-torsion elements and C_* is a projective $\mathbb{Z}\pi$-chain complex with the following properties:*
(1) $H_0(C_*)=H_2(C_*)=\mathbb{Z}$,
(2) $H_1(C_*)=H_3(C_*)=H_4(C_*)=0$,
(3) *the first homological k-invariant of C_* is non-zero,*
then C_* is not chain-homotopy equivalent to the chain complex of any topological space. □

2. The bar and $\bar{W}$-constructions

In this section we will develop an equivariant contraction from the $\bar{W}$-construction developed by Eilenberg and MacLane in [2] and their bar construction. We will make extensive use of the Eilenberg–Zilber theorem as presented in [3]:

Theorem 2.1. *Let V and V be FD-complexes and let $\hat{f}:U\times V\to U\otimes V$; $\hat{g}:U\otimes V\to U\times V$ be defined by*

$$\hat{f}:(u_n\times v_n)=\sum_{i=1}^{n}\tilde{F}^{n-i}u\otimes F_0^i v,$$

$$\hat{g}:(u_i\otimes v_j)=\sum_{(\mu,\nu)}(-1)^{p(\mu,\nu)}D_{\nu_j}\cdots D_{\nu_1}u_i\times D_{\mu_i}\cdots D_{\mu_1}v.$$

Then the triple $(\hat{f},\hat{g},\hat{F})$ induces a contraction of $(U\times V)_N$ onto $U_N\otimes V_N$, where $\hat{\Phi}$ is defined by

$$\hat{\Phi}=0 \quad \text{in dimension } 0,$$

$$\hat{\Phi}_n=-(\hat{\Phi}_{n-1})'+(\hat{g}\circ\hat{f})'\circ D_0. \qquad \square$$

Remarks. (1) We have quoted the theorem here because we will be using the explicit formulas above for $\hat{f}$, $\hat{g}$, and $\hat{\Phi}$.

(2) See [2, §6] for a definition of the derived operators.
(3) The proof that $\hat{\Phi}^2 = 0$ is due to Weishu Shih in [8].
(4) The symbols F_0^i, $\bar{F}^{n-i}$ denote the 'first' and 'last' face operators given by

$$F_0^i = F_0 \cdots F_i, \qquad \bar{F}^{n-i} = F_{i+1} \cdots F_n.$$

(5) The symbol $\sum_{(\mu,\nu)}$ denotes the sum over all 'shuffels' of length (i,j), and $p(\mu,\nu)$ denotes the parity of a shuffel – see Section 5 of [2].
(6) The subscript N denotes the normalized chain complex – the quotient by the subcomplex generated by the degenerate elements.

Throughout the remainder of this section R will denote a fixed R-complex. Recall that the W-construction of R is defined to have chain rings

$$W_n(R) = R_{n-1} \otimes R_{n-2} \otimes \cdots \otimes R_0$$

and it has the important property that its twisted cartesian product $R \times_a W(R)$ is contractible, where $a : \bar{W}(R)_n \to R_{n-1}$ is the twisting function given by

$$a(r_{n-1} \otimes \cdots \otimes r_0) = r_{n-1}\varepsilon_{n-2}(r_{n-2}) \cdots \varepsilon_0(r_0)$$

where $\varepsilon_i = \varepsilon_0 \circ (F_0)^i$. Note that we are using the notation of Moore in exposés 12 and 18 of [1] for the W-construction and twisted cartesian products. Although this notation is not standard today, it has certain advantages in discussion to follow.

Definition 2.2. $\mathfrak{t} : R_i \times \bar{W}(R_i) \to R_{i-1} \times \bar{W}(R)_{i-1}$ is defined by

$$\mathfrak{t}(r_i \times w_i) = F_0(r_i) \cdot (a(w_i) - 1_{i-1}) \times F_0(w_i)$$

or (using the definition of a and of F_0 on $\bar{W}(R)$),

$$\mathfrak{t}(r_i' \times r_{i-1} \otimes \cdots \otimes r_0) = F_0(r_i') \cdot (r_{i-1} - 1_{i-1}) \times r_{i-2} \otimes \cdots \otimes r_0$$

if $\varepsilon(r_j) = 1$ for all j. □

The Eilenberg–Zilber theorem gives rise to a contraction $(\hat{f}, \hat{g}, \hat{\Phi}) : (R \times_a \bar{W}(R))_N \to R_N \otimes_a \bar{W}_N(R)$ and the Perturbation Lemma immediately implies that:

Proposition 2.3. *Let K be the complex $R_N \otimes W_N(R)$ equipped with the differential defined by $d_\otimes + \hat{f} \circ \mathfrak{t} \circ \mathcal{T}_\infty \circ \hat{g}$, where $d_\otimes$ is the usual differential and $\mathcal{T}_\infty = 1 + \sum_{i=1}^\infty (\hat{\Phi}\mathfrak{t})^i$. Then $(R_N, \bar{W}_N(R), K)$ is an acyclic constuction in the sense of exposé 4 of* [1] *with contracting chain-homotopy*

$$\mathfrak{s} = (a \times F_0)^{-1} \circ \mathcal{T}_\infty \circ \hat{g} : R_N \otimes W_N(R) \to \bar{W}_N(R) = 1 \otimes \bar{W}_N(R).$$

Remarks. (1) The proof of this result is similar to that of the twisted Eilenberg–Zilber theorem of [8] and [4] except that, in accordance with [1], we have written the product as fiber × base rather than base × fiber.

(2) The map $(a\times F_0)^{-1}:(R\times \bar{W}(R))_N\to \bar{W}_N(R)$ carries $r_n'\times r_{n-1}\otimes\cdots\otimes r_0$ to $r_n'\otimes r_{n-1}\otimes\cdots\otimes r_0$.

Proof. We begin by defining an element $r\times w$ of $R\times \bar{W}(R)$ to be of filtration degree $\leq n$ if w is a degeneration of an element of dimension $\leq n$. It is clear that $\mathfrak{t}$ lowers filtration degree by at least 1 and $d_{(R\times \bar{W}\times(R))}$ and $\hat{\Phi}$ do not raise it. The Purturbation Lemma then gives rise to a contraction $(\tilde{f},\tilde{g},\tilde{\Phi}):K\to(R\times_a W(R))_N=W_N(R)$ where:

(1) $\tilde{f}=\hat{f}\circ(1+\mathfrak{t}\circ\mathcal{T}_\infty\circ\Phi)$,

(2) $\tilde{g}=\mathcal{T}_\infty\circ g$,

(3) $\tilde{\Phi}=\mathcal{T}_\infty\circ\hat{\Phi}$.

Since (see exposé 13 of [1]) $(a\times F_0)^{-1}$ is the chain contraction of $W_N(R)$, it follows that $\tilde{f}\circ(a\times F_0)^{-1}\circ g$ is a chain contraction of K. But $f(1\times w)=1\otimes w$ since $\Phi(1\times w)$ is degenerate and $\mathcal{T}_\infty$ and $\mathfrak{t}$ maps norms to norms. This completes the proof. □

We will use this result to connect the W-construction with the bar construction. Our notation for the bar construction will coincide with that of [1]. Recall that the unreduced bar construction $\mathscr{B}(R)=R\otimes\bar{\mathscr{B}}(R)$ is contractible via the chain contraction $s:R\otimes\bar{\mathscr{B}}(R)\to 1\otimes\bar{\mathscr{B}}(R)$ that maps $r\otimes[v_1|\cdots|v_k]$ to $1\otimes[r|v_1\cdots|v_k]$. Note that the inverse $s^{-1}:1\otimes\bar{\mathscr{B}}(R)\to R\otimes\bar{\mathscr{B}}(R)$ is well defined. By abuse of notation we will often want to regard it as a map s^{-1}: $\bar{\mathscr{B}}(R)\to R\otimes\bar{\mathscr{B}}(R)$.

In [2] Eilenberg and MacLane defined a homomorphism of DGA-algebras $\mathfrak{g}:\bar{\mathscr{B}}(R_N)\to\bar{W}_N(R)$ via

$$\mathfrak{g}=(a\times F_0)^{-1}\circ\hat{g}\circ(1\otimes\mathfrak{g})\circ s^{-1}$$

(see Lemma 19.2 in [2]) – note that this definition is *inductive*. It is completed by defining $\mathfrak{g}(r)$ to be $r\otimes 1\cdots\otimes 1$ (the number of 1's equals the dimension of r). Our main result is:

Theorem 2.4. *There exists a chain map* $\mathfrak{f}:\bar{W}_N(R)\to\bar{\mathscr{B}}_N(R_N)$ *and a chain-homotopy* $\psi:\bar{W}_N(R)\to\bar{W}_N(R)$ *such that* $(\mathfrak{f},\mathfrak{g},\psi):\bar{W}_N(R)\to\bar{\mathscr{B}}_N(R_N)$ *is a contraction. The maps are defined inductively by*

(1) $\mathfrak{f}(r\otimes 1\otimes\cdots\otimes 1)=[r]$ *for* $r\in R$,

(2) $\mathfrak{f}=s\circ(1\otimes\mathfrak{f})\circ\hat{f}\circ\mathfrak{t}\circ\mathcal{T}\circ\iota$, *where* $\iota:\bar{W}_N(R)\to(R\times W(R))_N$ *maps* $w\in\bar{W}_N(R)$ *to* $1\times w$,

(3) $\psi=0$ *in dimension* 0 *and, in higher dimensions* $\psi(w)=-\mathfrak{s}\circ(w+\psi(dw))$.

Remarks. (1) Recall that $\mathfrak{s}$ is defined in 2.3.

(2) Note that, in general, the map $\mathfrak{f}$ doesn't *preserve products*. In [2] Eilenberg and Maclane proved that $\mathfrak{g}$ preserves products.

(3) Note that if R has a group action on it, then the maps $\mathfrak{f}, \mathfrak{g}, \psi$ *preserve* that action. This implies that the bar and W-construction are equivariantly chain-homotopy equivalent. It also implies that the Eilenberg–MacLane model of Eilenberg–MacLane spaces (defined in [2] semi-simplically) is equivariantly homotopy equivalent to the model due to Milgram in [7].

Proof. The proof is divided into several parts:

(1) $\mathfrak{f}$ is a chain-map and ψ, as defined inductively above, is a chain-homotopy from $\mathfrak{g} \circ \mathfrak{f}$ to 1. First, suppose that the inductive definition of ψ was

$$\psi(w) = \mathfrak{s} \circ \imath(\mathfrak{g} \circ \mathfrak{f}(w) - w - \psi(dw)).$$

Then the statement would follow immediately from 2.3 in the present paper and the proof of theorem 1 in exposé 2 of [1]. We must show that $\mathfrak{s} \circ \imath \circ \mathfrak{g} = 0$. But

$$\begin{aligned}
&\mathfrak{s} \circ \imath \circ \mathfrak{g}\\
&= (a \times F_0)^{-1} \circ \mathscr{T}_\infty \circ \hat{g} \circ \imath \circ (a \times F_0)^{-1} \circ \hat{g} \circ (1 \otimes \mathfrak{g}) \circ s^{-1}\\
&= (a \times F_0)^{11} \circ \mathscr{T}_\infty \circ (a \times F_0)^{-1} \circ \hat{g} \circ \cdots + (a \times F_0)^{-1} \circ \left(1 + \sum_{i=1}^{\infty} (\hat{\Phi}\mathfrak{t})^i\right) \circ (a \times F_0)^{-1} \circ g \cdots\\
&= (a \times F_0)^{-1} \circ (a \times F_0)^{-1} \circ \hat{g} \cdots + (a \times F_0)^{-1} \circ \left\{1 + \sum_{i=1}^{\infty} (\mathfrak{t}\hat{\Phi})^i\right\} \circ (a \times F_0)^{-1} \circ g \cdots\\
&= 0 + (a \times F)^{-1} \circ \hat{\Phi} \circ \left\{1 + \sum_{i=1}^{\infty} (\mathfrak{t}\hat{\Phi})^i\right\} \circ \hat{g} \circ \cdots
\end{aligned}$$

(because $((a \times F_0)^{-1})^2$ is degenerate and because $\mathfrak{t} \circ (a \times F_0)^{11}(a \times b) = a \times b - 1 \times b$ and $\hat{\Phi}$ maps all elements of the form $1 \times b$ into degenerates), $= 0$ (because $\hat{\Phi} \circ \hat{g} = 0$).

(2) $\mathfrak{f} \circ \mathfrak{g} = 1 : \bar{\mathscr{B}}(R) \to \bar{\mathscr{B}}(R)$. This follows by induction on the simplical dimension of elements of $\bar{\mathscr{B}}(R)$ and direct computation. Clearly, it is true for elements of simplical dimension 1. In general

$$\begin{aligned}
\mathfrak{f} \circ \mathfrak{g} &= s \circ (1 \otimes \mathfrak{f}) \circ \hat{f} \circ \mathfrak{t} \circ \mathscr{T}_\infty \circ \imath \circ (a \times F_0)^{-1} \circ \hat{g} \circ (1 \otimes \mathfrak{g}) \circ s^{-1}\\
&= s \circ (1 \otimes \mathfrak{f}) \circ \hat{f} \circ \left\{1 + \sum_{i=1}^{\infty} (\mathfrak{t}\hat{\Phi})^i\right\} \circ \mathfrak{t} \circ (a \times F_0)^{-1} \circ \hat{g} \circ (1 \otimes \mathfrak{g}) \circ s^{-1}\\
&= s \circ (1 \otimes \mathfrak{f}) \circ \hat{f} \circ \left\{1 + \sum_{i=1}^{\infty} (\mathfrak{t}\hat{\Phi})^i\right\} \circ \hat{g} \circ (1 \otimes \mathfrak{g}) \circ s^{-1}
\end{aligned}$$

(because $\mathfrak{t} \circ (a \times F_0)^{-1}(a \times b) = a \times b - 1 \times b$ and $\hat{\Phi}(1 \times b)$ is degenerate and $(1 \otimes \mathfrak{f}) \circ f(1 \times b) = 1 \oplus \mathfrak{f}(b)$, which is mapped to 0 by s)

$$\begin{aligned}
&= s \circ (1 \otimes \mathfrak{f}) \circ \hat{f} \circ \hat{g} \circ (1 \otimes \mathfrak{g}) \circ s^{-1} \qquad \text{(because } \hat{\Phi} \circ \hat{g} = 0)\\
&= s \circ (1 \otimes \mathfrak{f}) \circ (1 \otimes \mathfrak{g}) \circ s^{-1} \qquad \text{(because } \hat{f} \circ \hat{g} = 1)\\
&= s \circ s^{-1} = 1 \qquad \text{(by the inductive hypothesis).}
\end{aligned}$$

(3) $\psi \circ \mathfrak{g}=0$. This follows by induction on the dimension. It is clearly true in dimension 0 because $\psi=0$. In higher dimensions

$$\begin{aligned}\psi(\mathfrak{g}(w)) &= -\mathfrak{s}\circ\iota(\mathfrak{g}(w)+\psi(d\mathfrak{g}(w)))\\ &= -\mathfrak{s}\circ\iota(\mathfrak{g}(w)+\psi(\mathfrak{g}(dw))) \qquad \text{(because } \mathfrak{g} \text{ is a chain-map)}\\ &= -\mathfrak{s}\circ\iota(\mathfrak{g}(w)) \qquad \text{(by the inductive hypothesis)}\\ &= 0 \qquad \text{(because } \mathfrak{s}\circ\iota\circ\mathfrak{g}=0 \text{ by part (1) of the proof of this theorem).}\end{aligned}$$

(4) $\mathfrak{f}\circ\mathfrak{s}=s\circ(1\otimes\mathfrak{f})$. This follows by direct computation:

$$\begin{aligned}\mathfrak{f}\circ\mathfrak{s} &= s\circ(1\otimes\mathfrak{f})\circ\hat{f}\otimes\mathfrak{t}\circ\mathscr{T}_\infty\circ\iota\circ(a\times F_0)^{-1}\circ\mathscr{T}_\infty\circ\hat{g}\\ &= s\circ(1\otimes\mathfrak{f})\circ\hat{f}\circ\left(1+\sum_{i=1}^{\infty}(\mathfrak{t}\hat{\Phi})^i\right)\circ\mathfrak{t}\circ(a\times F_0)^{-1}\circ\mathscr{T}_\infty\circ\hat{g}\\ &= s\circ(1\otimes\mathfrak{f})\circ\hat{f}\circ\left(1+\sum_{i=1}^{\infty}(\mathfrak{t}\hat{\Phi})^i\right)\circ\mathscr{T}_\infty\circ\hat{g} \qquad \text{(see the proof of (2))}\\ &= s\circ(1\otimes\mathfrak{f})\circ\hat{f}\circ\left(1+\sum_{i=1}^{\infty}(\mathfrak{t}\hat{\Phi})^i+(\hat{\Phi}\mathfrak{t})^i\right)\circ\hat{g}\end{aligned}$$

(because $\hat{\Phi}^2=0$, so all cross-terms in the composite vanish)

$$\begin{aligned}&= s\circ(1\otimes\mathfrak{f})\circ\hat{f}\circ\hat{g} \qquad \text{(because } \hat{\Phi}\circ\hat{g}=0 \text{ and } \hat{f}\circ\hat{\Phi}=0)\\ &= s\circ(1\otimes\mathfrak{f}) \qquad \text{(because } \hat{f}\circ\hat{g}=1).\end{aligned}$$

(5) $\mathfrak{f}\circ\psi=0$. This follows from statement (4) above.

$$\begin{aligned}\mathfrak{f}\circ\psi(w) &= \mathfrak{f}\circ\mathfrak{s}\circ\iota(*)\\ &= s\circ(1\otimes\mathfrak{f})\circ\iota(*)=0\end{aligned}$$

because $s(1\otimes *)$ is *degenerate*. □

Lemma 2.7. *There exists a contraction* $(\mathfrak{f}_2,\mathfrak{g}_2,\psi_2):\bar{\mathscr{B}}(\bar{W}_N(R))\to\bar{\mathscr{B}}^2(R_N)$ *where:*

(1) $\mathfrak{g}_2=\bar{\mathscr{B}}(\mathfrak{g})$,

(2) $\mathfrak{f}_2=\bar{\mathscr{B}}(\mathfrak{f})\circ\left(1+\sum_{i=2}^{\infty}(d_s\bar{\psi})^i\right)$,

(3) $\psi_2=\left(1+\sum_{i=1}^{\infty}(\bar{\psi}d_s)^i\right)\circ\bar{\psi}$.

d_s *is the simplical component of the boundary operator in* $\bar{\mathscr{B}}(W_N(R))$ *and* $\bar{\psi}$ *is defined inductively by:*

(A) $\bar{\psi}[\,]=0$,

(B) $\bar{\psi}[a\,|\,u]=-[\psi(a)\,|\,u]+(-1)^{\dim[a]}[\mathfrak{g}\circ\mathfrak{f}(a)\,|\,\bar{\psi}(u)]$.

Proof. First, note that

$$(\bar{\mathscr{B}}(\mathfrak{f}), \bar{\mathscr{B}}(\mathfrak{g}), \bar{\psi}) : \bar{\mathscr{B}}(W_N(R)) \to \bar{\mathscr{B}}^2(R_N)$$

would be a contraction if the boundary operators in $\bar{\mathscr{B}}(\bar{W}_N(R))$ and $\bar{\mathscr{B}}^2(R_N)$ consisted only of their *residual components* (see the proof of Theorem 12.1 in [2]).

Unfortunately, the map $\mathfrak{f}$ doesn't generally preserve products so that $\bar{\mathscr{B}}(\mathfrak{f})$ is usually not a *chain-map* with the full boundary operator in the bar construction. We will remedy this situation by using the Perturbation Lemma with t set equal to d_s – the simplical component of the boundary of $\bar{\mathscr{B}}(\bar{W}_N(R))$. We filter the elements of $\bar{\mathscr{B}}(\bar{W}_N(R))$ by their *simplical dimension.*

We get

$$f' = \bar{\mathscr{B}}(\mathfrak{f}) \circ \left(1 + \sum_{i=2}^{\infty} (d_s\bar{\psi})^i\right),$$

$$g' = \left(1 + \sum_{i=1}^{\infty} (\bar{\psi} d_s)^i\right) \circ \bar{\mathscr{B}}(\mathfrak{g}),$$

$$\psi' = \left(1 + \sum_{i-1}^{\infty} (\bar{\psi} d_s)^i\right) \circ \bar{\psi}.$$

Since the map $\mathfrak{g}$ was proved in [2] to preserve products, it follows that $\bar{\mathscr{B}}(\mathfrak{g})$ commutes with d_s so that the formula for g' becomes $\bar{\mathscr{B}}(\mathfrak{g})$ since it is not hard to see that $\bar{\psi} \circ \bar{\mathscr{B}}(\mathfrak{g}) = 0$. The only thing wrong with this procedure is that the Purturbation lemma induces a second boundary operator on $\bar{\mathscr{B}}^2(R)$ and this might not agree with its true boundary. The second boundary is given by

$$d' = d_r + \bar{\mathscr{B}}(\mathfrak{f}) \circ d_s \circ \left(1 + \sum_{i=1}^{\infty} (\bar{\psi} d_s)^i\right) \circ \bar{\mathscr{B}}(\mathfrak{g})$$

where d_r is the *residual boundary* (see [2]). By the same argument as was used for g':

$$\bar{\mathscr{B}}(\mathfrak{f}) \circ d_s \circ \left(1 + \sum_{i=1}^{\infty} (\bar{\psi} d_s)^i\right) \circ \bar{\mathscr{B}}(\mathfrak{g}) = \bar{\mathscr{B}}(\mathfrak{f}) \circ d_s \circ \bar{\mathscr{B}}(\mathfrak{g})$$

$$= \bar{\mathscr{B}}(\mathfrak{f}) \circ \bar{\mathscr{B}}(\mathfrak{g}) \circ d_s = d_s$$

so that $d' = d_r = d_s$ which is the *true boundary operator* in $\bar{\mathscr{B}}^2(R_N)$. □

An inductive application of 2.6 and 2.7 gives:

Corollary 2.8. *For any R-complex, R, and any positive integer k, there exists a contraction*

$$(\mathfrak{f}_k, \mathfrak{g}_k, \psi_k) : \bar{W}_N^k(R) \to \bar{\mathscr{B}}^k(R_N). \quad \square$$

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