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THE EQUIVARIANT STRUCTURE OF EILENBERG-MAC LANE SPACES. I. THE Z-TORSION FREE CASE

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ABSTRACT. The purpose of this paper is to continue the work begun in [**7**]. That paper described an obstruction theory for topologically realizing an (equivariant) chain-complex as *the* equivariant chain-complex of a CW-complex. The obstructions essentially turned out to be homological k-invariants of Eilenberg-Mac Lane spaces and the key to their computation consists in developing tractable models for the chain-complexes of these spaces. The present paper constructs such a model in the $\mathbf{Z}$-torsion free case. The model is sufficiently simple that in some cases it is possible to simply read off homological k-invariants, and thereby derive some topological results.

Introduction. Recall the *bar-construction* $\overline{\mathcal{B}}(*)$ of Eilenberg and Mac Lane—see [**2**]. If M is an abelian group it is a well-known fact that the chain-complex of an Eilenberg-Mac Lane space $K(M,n)$ is chain-homotopy equivalent to n-fold iterated bar construction $\overline{\mathcal{B}}^n(\mathbf{Z}M)$ (which we will denote as $A(M,n)$). Our main result is

THEOREM. *There is a functor $\mathcal{A}$ from torsion free abelian groups to torsion-free DGA-algebras, and a natural transformation $e\colon \overline{\mathcal{B}}(\mathbf{Z}M) \to \mathcal{A}(M)$ with the following properties*:

(i) *e is a homology equivalence*;

(ii) *$\mathcal{A}(M)$ is finitely generated in each dimension if M is finitely generated.*

REMARKS. 1. This is essentially Theorem 1.5.

2. This immediately implies the existence of a natural transformation $A(M,n) \to \overline{\mathcal{B}}^{n-1}(\mathcal{A}(M))$ that is a homology equivalence.

Before we state our next result we recall the definition of the DGA-algebra $U(M,2)$ given in [**3**, §18]:[1] For all integers $t \geq 1$ $U(M,2)_{2t-1} = 0$ and $U(M,2)_{2t}$ is generated, as an abelian group, by symbols $\gamma_t(m)$ for all $m \in M$ and these symbols satisfy the relations: $\gamma_0(m) = 1 \in U(M,n)_0 = \mathbf{Z}$; $\gamma_\alpha(m) \bullet \gamma_\beta(m) = (\alpha+\beta)!/\alpha!\beta!\gamma_{\alpha+\beta}(m)$, for all $m \in M$ and $\alpha, \beta \geq 0$;

$$\gamma_t(m_1+m_2) = \sum_{\alpha+\beta=t} \gamma_\alpha(m_1) \bullet \gamma_\beta(m_2); \quad \gamma_t(rm) = r^t\gamma_t(rm),$$

for all $m \in M$ and $r \in \mathbf{Z}$.

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[1]This DGA-algebra was denoted $\Gamma(M)$ there but our notation is standard today.

For instance $U(M,2)_4 = \Gamma(M)$ and $U(M,2)_{2t}$ is a t-fold *symmetric power* of M—the submodule of M^t generated by elements of the form $m \otimes \cdots \otimes m$ (t factors) for all $m \in M$. Let $\Omega(M)$ denote the following pull-back (or fibered product):

$$\begin{array}{ccc} & & \Gamma(M) \\ & & \downarrow g \\ M & \xrightarrow[p]{} & M/2M \end{array}$$

Note that there exists a natural projection $\mathcal{F}\colon \Omega(M) \to M$. The complex $\mathcal{A}(M)$ defined in §1 has the property that its 1-dimensional chain module is *precisely* $\Omega(M)$. This implies

COROLLARY 1. *A splitting of $\mathcal{F}\colon \Omega(M) \to M$ naturally determines a DGA-algebra map $U(M,2) \to \overline{\mathcal{B}}\mathcal{A}(M)$ which is a homology equivalence.*

REMARK. Such a splitting exists if $M/2M = 0$—e.g. if M is a module over $\mathbf{Z}[1/2]$.

PROOF. The hypothesis implies that $\Omega(M) = M \oplus \Gamma(M)$, so that $\overline{\mathcal{B}}(\mathcal{A}(M))_{2k}$ has a *direct summand* equal to M^k. We map $U(M,2)$ to $\overline{\mathcal{B}}\mathcal{A}(M)$ via the map that sends $\gamma_t(m) \in U(M,2)_{2t}$ to $[m|_2 \cdots |_2 m] \in \overline{\mathcal{B}}\mathcal{A}(M)_{2k}$ (t copies of m). This map induces an isomorphism of homology. This statement follows from the *proof* of Theorem 21.1 on p. 117 of [**3**]. Theorem 18.1 (of [**3**]) and the Künneth formula imply that the homology of $A(M,2)$ is $\mathbf{Z}$-torsion free. This implies that the map π_* on p. 117 of [**3**] is an isomorphism and the conclusion follows. □

If Z_* is a projective $\mathbf{Z}\pi$-resolution of $\mathbf{Z}$ then $e \otimes 1\colon A(M,1) \otimes Z_* \to \mathcal{A}(M) \otimes Z_*$ is a *chain-homotopy equivalence.* This implies that we can use $\mathcal{A}(M)$ to compute the *equivariant chain-complexes* and some of the *homological k-invariants*[2] of Eilenberg-Mac Lane spaces—these turn out to be significant in *topological applications* of this theory:

COROLLARY 2. *Let M be a $\mathbf{Z}$-torsion free $\mathbf{Z}\pi$-module and let Z be a projective $\mathbf{Z}\pi$-resolution of $\mathbf{Z}$. The first homological k-invariant of $A(M,n) \otimes Z$ is*

(a) $\alpha^*(x) \in \operatorname{Ext}^3_{\mathbf{Z}\pi}(M, \Gamma(M)) = H^3(\pi, \operatorname{Hom}_{\mathbf{Z}}(M, \Gamma(M)))$, *if* $n = 2$;

(b) $\beta_*\alpha^*(x) \in \operatorname{Ext}^3_{\mathbf{Z}\pi}(M, M/2M) = H^3(\pi, \operatorname{Hom}_{\mathbf{Z}}(M, M/2M))$, *if* $n \geq 2$;

where $\alpha\colon M \to M/2M$ and $\beta\colon \Gamma(M) \to M/2M$ are the projections and $x \in \operatorname{Ext}^3_{\mathbf{Z}\pi}(M/2M, \Gamma(M))$ is the class represented by the following 3-fold extension of $\mathbf{Z}\pi$-modules:

$$0 \to \Gamma(M) \xrightarrow[①]{} M \otimes M \xrightarrow[②]{} M \otimes M \xrightarrow[③]{} \Gamma(M) \xrightarrow[④]{} M/2M \to 0$$

where map 1 is diagonal inclusion ($\gamma(m) \to m \otimes m$), map 2 is antisymmetrization ($m_1 \otimes m_2 \to m_1 \otimes m_2 - m_2 \otimes m_1$), map 3 is symmetrization ($m_1 \otimes m_2 \to \gamma(m_1) + \gamma(m_2) - \gamma(m_1 + m_2)$) and map 4 sends $\gamma(m)$ to the class of m. □

REMARKS. 1. Recall that $\Gamma(M)$ is Whitehead's "universal quadratic functor".

[2]Recall that homological k-invariants are a *homological analogue* of *topological k-invariants*—a chain-complex whose homological k-invariants *all vanish* is chain-homotopy equivalent to a direct sum of suspended projective resolutions of its homology modules. For a discussion of homological k-invariants see [**4**].

2. From this result it is *immediately clear* that the first homological k-invariant of $A(M,2)$ is a 2-*torsion element.*

3. This corollary follows from the description of the low-dimensional structure of $\mathcal{A}(M)$ in the discussion that precedes 1.1.

4. The formula $\mathrm{Ext}^3_{\mathbf{Z}\pi}(M,\Gamma(M)) = H^3(\pi, \mathrm{Hom}_{\mathbf{Z}}(M,\Gamma(M)))$ makes use of the main result of [**6**].

5. Here is an example of a module M for which this invariant is *nonzero* (see [**5**] for a proof): $\pi = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ on generators s and t, $M = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ and s and t act via right multiplication by the matrices

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \text{respectively.}$$

PROOF. Recall the definition of $\Omega(M)$ given in Remark 3 following Theorem 1. We can define the *symmetrization map* $\mathcal{S}\colon M \otimes M \to \Omega(M)$—it sends $m_1 \otimes m_2$ to $\gamma(m_1) + \gamma(m_2) - \gamma(m_1 + m_2) \in \ker g \subset \Omega(M)$. The kernel of this map is $\Lambda^2(M)$ (since M is $\mathbf{Z}$-torsion free) and the cokernel is M. The projection to the cokernel $\Omega(M) \to M$ is denoted $\mathcal{F}$. We can, consequently, define maps:

$A(M,1)_1 \to \Omega(M)$, sending $[m]$ to the class of $(m, \gamma(m))$;

$A(M,1)_2 \to M \otimes M$, sending $[m_1|m_2]$ to $m_1 \otimes m_2$;

and it is not hard to see that this is a *chain-map* from the 2-skeleton for $A(M,1)$ to the chain-complex C_*, where $C_1 = \Omega(M)$ and $C_2 = M \otimes M$ and where the boundary map is $\mathcal{S}$. Furthermore this map induces isomorphisms in homology in dimensions 1 and 2. This implies the corollary. □

This has immediate consequences in the study of the *Steenrod problem* and the related question of when *chain-complexes* are *topologically realizable.* Let $\tilde{K}(\pi,1)$ denote the *universal covering space* of a $K(\pi,1)$. The first result of the present paper, coupled with the theory of realizations of chain-complexes presented in [**7**] implies

COROLLARY 3. *Let X be a topological space with $\pi_1(X) = \pi$, $H_i(X;\mathbf{Z}\pi) = M$, a $\mathbf{Z}$-torsion free $\mathbf{Z}\pi$-module, and with $H_{i+1}(X;\mathbf{Z}\pi) = H_{i+2}(X;\mathbf{Z}\pi) = 0$ for some $i \geq 2$ and suppose that $H_j(X;\mathbf{Z}\pi) = 0$ for all $2 \leq j < i$. If the first k-invariant of X is 0 then the k-invariant of X in $H^{i+3}(K(M,i) \times_\pi \tilde{K}(\pi,1); H_{i+2}(K(M,i)) = H^{i+3}(K(M,i) \times_\pi \tilde{K}(\pi,1); V) = H^3(\pi, \mathrm{Hom}_{\mathbf{Z}}(M,V))$ must be equal to*

$\alpha^*(x)$ defined in statement (a) *of Corollary* 2 *if $i = 2$ (here $V = \Gamma(M)$);*

$\beta_*\alpha^*(x)$ defined in statement (b) *of Corollary* 2 *if $i > 2$ (here $V = M/2M$).* □

REMARKS. We take the cartesian product of $K(M,i)$ with $\tilde{K}(\pi,1)$ and equip the result with the *diagonal π-action* so that we will have a space upon which π acts *freely.*

COROLLARY 4. *Let C be an $i+3$-dimensional projective $\mathbf{Z}\pi$-chain-complex for some $i > 2$ with*

1. *$H_0(C) = \mathbf{Z}$ and $H_i(C) = M$, a $\mathbf{Z}$-torsion free $\mathbf{Z}\pi$-module*;
2. *$H_j(C) = 0$ for all $2 \leq j \leq i$.*

Then C_* is topologically realizable iff the element $e \in H^{i+3}(C^+, M/2M)$ vanishes where e is defined as follows:

Let $\mathfrak{M}$ be the $\mathbf{Z}$-free $\mathbf{Z}\pi$-chain-complex

$$0 \to \Gamma(M) \underset{\textcircled{1}}{\longrightarrow} M \otimes M \underset{\textcircled{2}}{\longrightarrow} M \otimes M \underset{\textcircled{3}}{\longrightarrow} \Omega(M) \to 0$$

and regard it as a resolution of M. Let $\alpha\colon C^+ \to \Sigma^i \mathfrak{M}$ be the unique chain-homotopy class of chain maps inducing the identity map in homology in dimension i. Then e is the cocycle that results from forming the composite

$$C_{i+3} \xrightarrow[\alpha_{i+3}]{} \Gamma(M) \underset{\textcircled{4}}{\longrightarrow} M/2M. \quad \square$$

REMARKS. 1. Here C^+ is a desuspension of the algebraic mapping cone of the unique (up to a chain-homotopy) chain-map $C \to Z$ induced by the augmentation $\varepsilon\colon C \to \mathbf{Z}$, where Z is a projective resolution of $\mathbf{Z}$ over $\mathbf{Z}\pi$. C^+ is uniquely determined up to an isomorphism (since homotopic maps give rise to isomorphic algebraic mapping cones).

2. The circled maps 1, 2, 3 and 4 have the same significance here as they do in the preceding theorem and $\Omega(M)$ has the meaning it was given in the discussion preceding Corollary 1.

3. Since α is unique up to a chain-homotopy, the class $e \in H^{i+3}(C^+, M/2M)$ is *uniquely defined* and only depends upon C.

4. See §2 for the proof.

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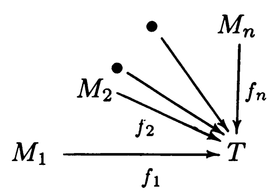
1. Proof of the main result. Consider the fibered product P, formed with respect to the following diagram:



Let the canonical maps from P to the M_i be $\tilde{f}_i\colon P \to M_i$—these have the well-known property that $f_i \circ \tilde{f}_i = f_j \circ \tilde{f}_j$ for all i and j. We will make use of the following well-known properties of such fibered products in the sequel:

PROPERTY 1. The canonical map $c\colon P \to T$ has the property that

$$\ker c = \prod_{i=1}^{n} \ker f_i.$$

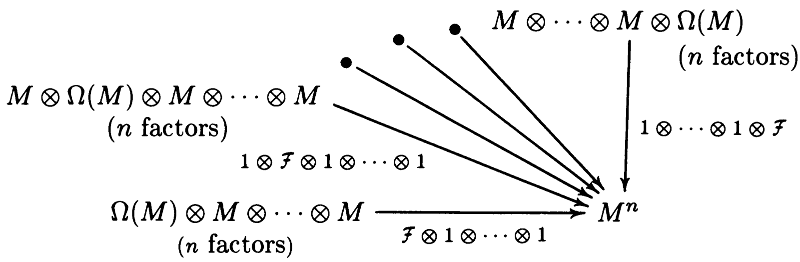
PROPERTY 2. Let V be a $\mathbf{Z}$-module and $g_i\colon V \to M_i$ is a set of homomorphisms such that $f_i \circ g_i = f_j \circ g_j$. Then the canonical map $h\colon V \to P$ such that $g_i = \tilde{f}_i \circ h$ has the property that $\ker h = \bigcap_{i=1}^{n} \ker g_i$.

The remainder of this section will be spent extending the chain-map defined in the proof of Corollary 2 to the higher dimensions of $A(M, 1)$.

DEFINITION 1.1. Define $\Omega_n(M)$ to be

1. $\mathbf{Z}$ if $n = 0$;
2. $\Omega(M)$ if $n = 1$;

3. The fibered product of the diagram:



if $n > 1$. □

REMARKS. 1. In the diagram above there are n objects mapping to M^n—and M^n denotes an n-fold tensor product (over $\mathbf{Z}$) of M with itself.

2. Consider the map $\mathcal{S}_n \colon M^n \to \Omega_{n-1}(M)$ defined to be $\mathcal{S} \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \mathcal{S} \otimes 1 \otimes \cdots \otimes 1 + \cdots + (-1)^n 1 \otimes \cdots \otimes 1 \otimes \mathcal{S}$ ($n-2$ factors equal to the identity map in each term). Property 2 of a fibered product implies that the kernel of this map is $\Lambda^2(M) \otimes M \otimes \cdots \otimes M \cap M \otimes \Lambda^2(M) \otimes M \otimes \cdots \otimes M \cap \cdots \cap M \otimes \cdots \otimes M \otimes \Lambda^2(M)$ ($n-1$ factors in each term) $= \Lambda^n(M)$.

3. An element of $\Omega_n(M)$ will be denoted by $[(m_1, e_1)(m_2, e_2) \cdots (m_n, e_n)]$, where $m_i \in M$ and $e_i \in \Omega(M)$. The following facts are easily verified:

PROPOSITION 1.2. (a) $[(m_1, e_1)(m_2, e_2) \cdots (m_n, e_n)]$ *maps to* $m_1 \otimes \cdots \otimes m_n$ *under the canonical projection* $p_n \colon \Omega(M) \to M^n$;

(b) *in* $[(m_1, e_1)(m_2, e_2) \cdots (m_n, e_n)]$ *if any* $m_i = 0$ *then the values of the* e_j *for* $j \neq i$ *are not significant*;

(c) *the kernel of* p_n *is generated by elements of the form*

$$[(m_1, e_1)(m_2, e_2) \cdots (m_n, e_n)]$$

with $m_i = 0$ *for some* i *and the corresponding* e_i *equal to* $\mathcal{S}(m \otimes m')$ *for some* $m, m' \in M$;

(d) *any symbol* $[(m_1, e_1)(m_2, e_2) \cdots (m_n, e_n)]$ *with* $m_i = 0$ *for two distinct indices* i *represents the zero element of* $\Omega_n(M)$. □

PROPOSITION 1.3. *There exists a bilinear map* $b \colon \Omega_i(M) \otimes \Omega_j(M) \to \Omega_{i+j}(M)$ *that sends*

$$[(m_1, e_1)(m_2, e_2) \cdots (m_i, e_i)] \otimes [(m_{i+1}, e_{i+1})(m_{i+2}, e_{i+2}) \cdots (m_{i+j}, e_{i+j})]$$

to $[(m_1, e_1)(m_2, e_2) \cdots (m_{i+j}, e_{i+j})]$.

PROOF. Simply note that the fibered products with respect to the diagrams

$$\begin{array}{ccc} & & M^i \\ & & \downarrow \\ \Omega_i(M) & \longrightarrow & M^i \end{array} \quad \text{and} \quad \begin{array}{ccc} & & M^j \\ & & \downarrow \\ \Omega_j(M) & \longrightarrow & M^j \end{array}$$

are $\Omega_i(M)$ and $\Omega_j(M)$, respectively. These fibered products are also submodules of $\Omega_i(M) \oplus M^i$ and $\Omega_j(M) \oplus M^j$ so we can form the tensor product of these direct sums and project onto the summand $\Omega_i(M) \otimes M^j \oplus M^i \otimes \Omega_i(M)$. Now substituting the definitions of the $\Omega_i(M)$'s into this direct sum implies the existence of a linear map from $\Omega_i(M) \otimes M^j \oplus M^i \otimes \Omega_i(M)$ to $\Omega_{i+j}(M)$. □

This tensor product bilinear mapping implies that we can define an analogue to the *shuffle product* (in the bar construction) on the $\Omega_i(M)$'s—see [**2**].

PROPOSITION 1.4. *Define a chain-complex $\Omega_*(M)$ as follows:*

1. *$\Omega_*(M)_i = \Omega_i(M)$ as defined above;*
2. *the boundary map $\Omega_i(M) \to \Omega_{i-1}(M)$ is defined to be 0 if $i = 1$ and $S_n \circ p$ where S_n is defined in Remark 2 above and p is the canonical projection $\Omega_i(M) \to M^i$.*

Then the map $A(M,1) \to \Omega_*(M)$ that sends $[m_1|\cdots|m_n]$ to

$$[(m_1, \omega(m_1))(m_2, \omega(m_2)) \cdots (m_n, \omega(m_n))]$$

is a chain map. Furthermore it carries the shuffle product on the bar construction to that on $\Omega_*(M)$ and so defines a homomorphism of DGA-algebras. □

REMARKS. 1. This follows by a straightforward induction on n.

2. This map *is not a homology equivalence*—for instance property 2 of a fibered product implies that the cycle module $Z_i(\Omega_*(M)) = p^{-1}(\Lambda^n(M))$ and property 1 implies that $p^{-1}(0) = S^2(M) \otimes M \otimes \cdots \otimes M \oplus M \otimes S^2(M) \otimes M \otimes \cdots \otimes M \oplus \ldots$, where $S^2(M)$ is the image of S—the symmetric product of M.

The final step in computing the model for $A(M,1)$ consists in modifying this chain-complex giving a complex denoted $\mathcal{A}(M)$ so that the canonical map from $A(M,1) \to \mathcal{A}(M)$ becomes a *homology equivalence* and extending the shuffle product to $\mathcal{A}(M)$. The main result of this section is

THEOREM 1.5. *Let $\mathcal{A}(M)$ denote the following chain-complex:*

1. *$\mathcal{A}(M)_i = \Omega_i(M)$ if $i < 3$;*
2. *$\mathcal{A}(M)_i = \Omega_i(M) \oplus \bigoplus_{j=1}^{i-2} F_{ij}(M)$, where $F_{ij}(M) = M^j \otimes S^2(M) \otimes M^{i-2-j}$;*
3. *the boundary maps on the $\Omega_i(M)$-summands are identical to those on $\Omega_*(M)$;*
4. *the boundary map from $F_{ij}(M)$ to $\mathcal{A}(M)_{i-1}$ has its image in $\Omega_{i-1}(M)$. It sends $m_1 \otimes \cdots \otimes S(m_{j+1} \otimes m_{j+2}) \otimes \cdots \otimes m_i$ to*

$$[(m_1, \omega(m_1)) \cdots (0, S(m_{j+1} \otimes m_{j+2})) \cdots (m_n, \omega(m_n))].$$

Then the composite $A(M,1) \to \Omega_*(M) \subset \mathcal{A}(M)$ is a homology equivalence and $\mathcal{A}(M)$ can be given a DGA-algebra structure to make this map a DGA-algebra homomorphism.

REMARKS. Recall that $S^2(M)$ denotes the symmetric product of M—by abuse of notation we *identify* it with the image of $S: M^2 \to \Gamma(M)$ and its image in $\Omega(M)$. This is *possible* because M is **Z**-torsion free.

PROOF. Essentially we constructed $\mathcal{A}(M)$ so that $\mathcal{A}(M)_n/\partial(\mathcal{A}(M)_{n+1}) = M^n$. If we take that for granted for a moment it is not hard to see that the map $A(M,1) \to \mathcal{A}(M)$ described above *is* a homology equivalence.

Property 1 at the beginning of this section implies that the kernel of the canonical map $\Omega_i(M) \to M^i$ is $\bigoplus_{j=0}^{i-2} F_{ij}(M)$—note that here the summation starts from 0 rather than 1 in the definition of $\mathcal{A}(M)$. Essentially the boundary map from $\Omega_{i+1}(M)$ kills off *one copy* of $F_{ij}(M)$ and the terms $F_{ij}(M)$ in the definition of $\mathcal{A}(M)$ kill off the *remaining copies.*

All that remains to be done is to define the DGA-algebra structure $\mathcal{A}(M)$.

Claim. We may define $u^*u' = 0$, where $u \in F_{ij}(M)$, $u' \in F_{i'j'}(M)$.

This follows from the fact that 1.2(d) implies that the product of the *boundaries* of u and u' (which lie in $\Omega_*(M)$) must be 0.

In order to define z^*u, where $z \in \Omega_i(M)$ and $u \in F_{i'j'}(M)$ simply note that the tensor product of z by $\partial(u)$ (using the tensor product operation defined in 1.3) will be in the image of some $F_{i''j}(M)$ and this fact will not be altered by shuffling operations. The product z^*u is *uniquely defined* since the boundary operation on the $F_{ij}(M)$'s is *injective*. Note that the $F_{ij}(M)$'s will constitute an *ideal* in $\mathcal{A}(M)$ under this multiplication law. □

2. Proof of Corollary 4. The obstruction to topologically realizing a chain-complex in [**7**] are essentially obstructions to the existence of a chain-map from the original chain-complex to the chain-complex of a partial Postnikov tower.

The chain-complex of such a Postnikov tower will generally be an iterated twisted tensor product—except in the "stable range" where it will a be twisted *direct sum* (i.e. a desuspension of an algebraic mapping cone). This is the case in the *present result.* The chain-complex C is topologically realizable if and only if there exists a chain-map from C to $Z \oplus_\xi Z \otimes \mathfrak{M}$ inducing the identity map in homology in dimension i, where ξ is essentially the first homological k-invariant of C. (If ξ vanishes C is chain-homotopy equivalent to $Z \oplus C^+$.) Clearly such a chain-map will exist if and only if there exists a chain-map C^+ to $Z \otimes \mathfrak{M}$ (since C and $Z \oplus_\xi Z \otimes \mathfrak{M}$ are compatible chain-complex extensions of Z by C^+ and $Z \otimes \mathfrak{M}$, respectively). The obstruction to the existence of a chaim-map $C^+ \to Z \otimes \mathfrak{M}$ was described in [**7**] as the cocyle that results from taking the *following* composite:

$$C_{i+3} \xrightarrow[\partial]{} C_{i+2} \xrightarrow[\alpha_{i+2}]{} Z(\mathfrak{M}^{i+2})_{i+2} \to H_{i+2}(\mathfrak{M}^{i+2}) = \Gamma(M) \to M/2M$$

where we assume that the α-map has been constructed up to dimension $i+2$—but this is clearly equal to the cocyle described in the statement of the corollary. □

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