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COMPLEMENTS OF CODIMENSION-TWO SUBMANIFOLDS I: THE FUNDAMENTAL GROUP

BY

JUSTIN R. SMITH

Introduction and statement of results

This paper is the first in a series that will study the homotopy types of the complements of certain classes of codimension-two imbeddings of compact manifolds—in particular, this paper will study the groups that can occur as fundamental groups. The results of this paper apply equally to smooth, *PL*, or topological imbeddings and manifolds. All manifolds in this paper will be assumed to be *compact* and *connected* and all imbeddings will be assumed to be *locally-flat* and to carry the boundry of the imbedded manifold transversely to that of the ambient manifold.

This paper generalizes Kervaire's characterization of high-dimensional knot groups in [4].

The results in this paper formed part of my doctoral dissertation and I am indebted to my thesis advisor, Professor Sylvain Cappell, for having suggested this problem and for his guidance and criticism. I would also like to thank the referee for his helpful comments.

Before we can state the main result of this paper we need the following definition:

DEFINITION AND PROPOSITION 1. *Let M^m be a compact manifold and let*

$$w\colon \pi_1(M) \to \mathbf{Z}_2 = \{\pm 1\}$$

be a homomorphism and $\mathbf{Z}^w$ the $\mathbf{Z}\pi_1(M)$-module of twisted integers defined by w. If $x \in H^2(M, \mathbf{Z}^w)$ is any element, define

$$C(x, w) = \mathbf{Z}^w/(x \cap H_2(M; \mathbf{Z}\pi_1(M));$$

the cap product takes its values in $H_0(M; \mathbf{Z}^w \otimes \mathbf{Z}\pi_1(M)) = \mathbf{Z}^w$. If x' is the image of x under the change of coefficient homomorphism

$$H^2(M; \mathbf{Z}^w) \to H^2(M; C(x, w)),$$

then x' is in the image of the injection

$$H^2(\pi_1(M); C(x, w)) \to H^2(M; C(x, w))$$

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induced by the characteristic map of M. Let $G(x, w)$ *be the group extension of* $C(x, w)$ *by* $\pi_1(M)$ *defined by the inverse image of* x' *in* $H^2(\pi_1(M); C(x, w))$, *regarding this as the group of equivalence classes of extensions of* $C(x, w)$ *by* $\pi_1(M)$.

Remark. The statement that x' is in the image of the map in cohomology induced by the characteristic map of M will be proved in Section I.

Thus, for each twisted class $x \in H^2(M; \mathbf{Z}^w)$ we get a cyclic group $C(x, w)$ and a canonical imbedding $C(x, w) \to G(x, w)$ as a normal subgroup with quotient $\pi_1(M)$—henceforth we will identify $C(x, w)$ with its image in $G(x, w)$. Suppose $f: M^m \to V^{m+2}$ is an imbedding of compact manifolds. Then we define $w_f(g) = w_M(g)w_V(f_* g)$, where w_M and w_V are the orientation characters of M and V respectively and $g \in \pi_1(M)$. If $\chi_f \in H^2(M; \mathbf{Z}^{w_f})$ is the *Euler* class of f we will adopt the *abbreviated notation* $C_f = C(\chi_f, w_f)$, $G_f = G(\chi_f, w_f)$. Then the statement of our main theorem is:

THEOREM 2. *Let* M^m *and* V^{m+2} *be compact manifolds with* $m \geq 3$ *and suppose there exists an imbedding* $f: M^m \to V^{m+2}$ *that induces an isomorphism of fundamental groups and a surjection of second homotopy groups.*

Then a group G can be the fundamental group of the complement of an imbedding of M in V homotopic to f if and only if the following conditions hold:

(1) *G is finitely presented.*

(2) *There exists a homomorphism* $j: G \to G_f$, *split by a homomorphism* j_s *and such that*

(a) *if* $K = j^{-1}(C_f)$, *then K is the normal closure within itself of* $j_s(C_f)$ *and*

(b) $H_2(K, \mathbf{Z}) = 0$.

The proof will be given in Section II.

Remarks. (1) Note that G_f and its subgroup C_f only depend upon the homotopy class of f since they are determined by the Euler class and orientation character.

(2) Conditions (2)(a) and (2)(b) above imply that $H_1(K, \mathbf{Z}) = C_f$, the isomorphism being induced by j. This follows from the fact that, since C_f is *abelian*, the homomorphism $j|K: K \to C_f$ must factor through the map $K \to K/[K, K] = H_1(K, \mathbf{Z})$ and the fact that no cyclic group is isomorphic to a proper quotient of itself.

(3) In the case of a high-dimensional knot, $\pi_1(M) = \pi_1(V) = 0$ and $\chi_f = 0$ which implies that $C_f = G_f = \mathbf{Z}$. This and the remark above show that, in this case, Theorem 2 reduces to Kervaire's characterization of high-dimensional knot groups.

(4) The existence portion of the proof of Theorem 2 will construct an imbedding with complementary fundamental group any G satisfying the conditions above that is *concordant* to the imbedding f in the hypothesis—where two

imbeddings $f_0, f_1: M \to V$ are defined to be concordant if they are restrictions of an imbedding of $M \times I$ in $V \times I$ to $M \times 0$ and $M \times 1$, respectively. This shows that the groups that can occur as complementary fundamental groups are, in a sense, independent of the concordance class of the imbedding—they only depend upon the Euler class.

(5) Suppose f is an *orientable* map, i.e., f preserves orientation characters. Then we can define the Euler class of f as follows: Let $[M] \in H_m(M, \partial M; \mathbf{Z}^t)$, $[V] \in H_{m+2}(V, \partial V; \mathbf{Z}^t)$ be fundamental classes. Then there exists an $x \in H^2(V; \mathbf{Z})$ such that $x \cap [V] = f_*[M]$ and χ_f is then the image of x in $H^2(M; \mathbf{Z})$ under f^* (see [6]).

COROLLARY 3. *Let M^m, V^{m+2} be compact manifolds with $m \geq 3$, that are simply-connected and 2-connected, respectively, and suppose there exists an imbedding of M in V. Then a group can be the fundamental group of the complement of an imbedding of M in V if and only if it is a high-dimensional knot group.*

Proof. The conditions on M and V imply $H^2(V) = 0$ so that the Euler class of any imbedding of M in V is 0. The conclusion follows from Remark 3.

COROLLARY 4. *Let L_1^{2k-1}, L_2^{2k+1} be homotopy lens spaces of index n, i.e., they are quotients of spheres by $\mathbf{Z}_n$-actions, where n is an odd integer, and suppose there exists an imbedding of L_1 in L_2. Then a group G is the fundamental group of the complement of an imbedding of L_1 in L_2 if and only if:*

(1) *G is finitely presented;*
(2) *G is the normal closure of a single element x such that $G/(x^n)^G = \mathbf{Z}_n$, where $(x^n)^G$ is the normal closure of x^n;*
(3) $H_1((x^n)^G, \mathbf{Z}) = \mathbf{Z}$;
(4) $H_2((x^n)^G, \mathbf{Z}) = 0$.

Remark. It is not difficult to see that $H_1(G) = \mathbf{Z}$ and $H_2(G) = 0$ so that, by Kervaire's criteria, G is a high-dimensional knot group. Not all knot groups can occur in this manner though—all exponents of the Alexander polynomial must be multiples of n (this can be seen by regarding the Alexander polynomial as defining a presentation of the first homology module of the infinite cyclic covering of the complement).

Proof. In [2] Cappell and Shaneson have completely characterized locally-flat codimension-two imbeddings of homotopy lens spaces (in terms of invariant spheres under $\mathbf{Z}_n$-actions) and their results imply that any such imbedding induces an *isomorphism* of fundamental groups and has a Euler class that is a *unit* in $\mathbf{Z}_n$. This implies that in the statement of Theorem 2, $G_f = \mathbf{Z}$ and C_f is the subgroup $n \cdot \mathbf{Z}$. If G satisfies the conditions in the corollary, $G/[G, G]$ will be a cyclic group containing a copy of $\mathbf{Z}$ since it will contain a copy of $H_1((x^n)^G)$; so $G/[G, G] = \mathbf{Z}$ and this defines the map j in Theorem 2. The splitting j_s carries 1 in $\mathbf{Z}$ to $x \in G$ and the remaining conditions follow.

Conversely, if G satisfies the conditions of Theorem 2 it is only necessary to verify that G is the normal closure of an element whose nth power normally generates $K = (x^n)^G$.

But the image of $1 \in \mathbf{Z} = G_f$ under the splitting j_s has these properties.

I. Properties of the fundamental group

Proof of Proposition 1. Let $\Lambda^t = \mathbf{Z}^w \otimes \mathbf{Z}\pi_1(M)$ and consider the low-order exact sequence in cohomology induced by the universal covering space spectral sequence (see [1, chapter 15, Section 9])

$$0 \longrightarrow H^2(\pi_1(M); \mathbf{Z}^w) \xrightarrow{c^*} H^2(M; \mathbf{Z}^w) \xrightarrow{h} H^2(M; \Lambda^t)^f \longrightarrow \cdots$$

where the f in the term on the right denotes the submodule fixed by the elements of $\pi_1(M)$. Since $H_1(M; \mathbf{Z}\pi_1(M)) = 0$,

$$H^2(M, \Lambda^t) = \mathrm{Hom}_{\mathbf{Z}}\,(H_2(M; \Lambda^t), \mathbf{Z})$$

and we can regard h above as the dual of the Hurewicz homomorphism. Then c^* is the map induced by the characteristic map in cohomology and

$$H^2(M; \Lambda^t)^f = \mathrm{Hom}_{\mathbf{Z}}\left(H_2(M; \Lambda^t) \underset{\mathbf{Z}\pi_1(M)}{\otimes} \mathbf{Z}, \mathbf{Z}\right)$$

and h carries $x \in H^2(M; \mathbf{Z}^w)$ to the map $y \otimes n \to n(x \cap y)$, $y \in H^2(M; \Lambda^t)$, $n \in \mathbf{Z}$ (see [1, p. 28]). The change of coefficients from $\mathbf{Z}^w$ to $C(x, w)$ induces the following commutative exact diagram (of groups):

$$\begin{array}{ccccccc}
0 \to H^2(\pi_1(M); \mathbf{Z}^w) & \xrightarrow{c^*} & H^2(M; \mathbf{Z}^w) & \xrightarrow{h} & \mathrm{Hom}_{\mathbf{Z}}\left(H_2(M, \Lambda^t) \underset{\mathbf{Z}\pi_1(M)}{\otimes} \mathbf{Z}, \mathbf{Z}\right) & \to \cdots \\
\downarrow & & \downarrow p & & \downarrow & \\
0 \to H^2(\pi_1(M); C) & \xrightarrow{c'^*} & H^2(M; C) & \xrightarrow{h'} & \mathrm{Hom}_{\mathbf{Z}}\left(H_2(M, \Lambda^t) \underset{\mathbf{Z}\pi_1(M)}{\otimes} \mathbf{Z}, C\right) & \to \cdots
\end{array}$$

where $C = C(x, w)$. It is clear by the definition of $C(x, w)$ and the description of the map h above that $h'(x') = 0$ so that x' is in the image of c'^*.

LEMMA I.1. *Let M, w, and x be as in Proposition* 1. *If $S(\xi)$ is the total space of the unique (up to isomorphism) circle bundle with first Stiefel-Whitney class w and twisted Euler class x, then an isomorphism between $\mathbf{Z}^w$ and $H_0(M; \mathbf{Z}^w)$ induces an isomorphism between $C(x, w)$ and the cyclic subgroup F of $\pi_1(S(\xi))$ generated by the inclusion of a fiber. Furthermore, this isomorphism extends to a commutative exact diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & \pi_1(S(\xi)) & \longrightarrow & \pi_1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(x, w) & \xrightarrow{i} & G(x, w) & \longrightarrow & \pi_1(M) & \longrightarrow & 0
\end{array}$$

where i is the canonical inclusion of $C(x, w)$ in $G(x, w)$. In particular, by the 5*-lemma, $G(x, w)$ is isomorphic to $\pi_1(S(\xi))$.*

Proof. The first statement follows from the interpretation of the map h in the Thom-Gysin sequence in homology as cap-product with the Euler class (see [6, Chapter 5):

$$\cdots \longrightarrow H_2(M; \mathbf{Z}\pi_1(M)) \xrightarrow{h} H_0(M; \mathbf{Z}^w \otimes \mathbf{Z}\pi_1(M)) \longrightarrow H_1(S(\xi); \mathbf{Z}\pi_1(M)) \longrightarrow 0$$

where $H_1(S(\xi); \mathbf{Z}\pi_1(M))$ is the cyclic subgroup of $\pi_1(S(\xi))$ generated by the inclusion of a fiber, by Shapiro's lemma. The second statement follows from Proposition 11.4 of [7], which implies that there exists a circle fibration η over a space X such that $\pi_1(X)$ is isomorphic to $\pi_1(M)$ and, given any isomorphism between $\pi_1(X)$ and $\pi_1(M)$, there exists a unique homotopy class of maps $f\colon M \to X$ such that f_η^* is fiber homotopy equivalent to ξ. Furthermore, given any isomorphism $\pi_1(S(f_\eta^*)) \to \pi_1(S(\xi))$, compatible with f, there exists a unique homotopy class of fiber-homotopy equivalences inducing the isomorphism. The *proof* of 11.4 in [7] shows that, in this *universal fibration*, the image of the Euler class in $H^2(M; C)$ (where C is the cyclic group generated by the inclusion of the fiber) is the image of the class in $H^2(\pi_1(X), C)$ defining the group extension $0 \to C \to \pi_1(S(\eta)) \to \pi_1(X) \to 0$, under the map induced by the characteristic map of X. The conclusion now follows from the functoriality of Euler classes and group extension classes.

DEFINITION I.2. Let $f\colon M^m \to V^{m+2}$ be an imbedding of compact manifolds. Then there exists a homeomorphism $h\colon V \to V'$ such that a regular neighborhood of $hf(M)$ in V' is the total space of a 2-plane bundle ξ over M. An element of $\pi_1(V - f(M))$ will be called a meridian class if it is of the form $h_*^{-1}a'$, where a' is a fiber of the unit circle bundle associated to ξ. If $i_*\colon \pi_1(V - f(M)) \to \pi_1(V)$ is induced by inclusion, the kernel will be called the meridian subgroup.

The existence of h follows from the fact that f is locally-flat and the facts that TOP_2/PL_2 and PL_2/O_2 are contractible. Note that, in view of I.1 and I.2, if S is the boundary of a tubular neighborhood of $f(M)$ in V, we may identify G_f with $\pi_1(S)$ and C_f with the subgroup generated by inclusion of a fiber.

PROPOSITION I.3. *Let $f\colon M^m \to V^{m+2}$ be an imbedding of compact manifolds that induces a surjection of fundamental groups. Then the meridian subgroup of $\pi_1(V - f(M))$ is the normal closure, within itself, of a single meridian class.*

Proof. This follows by a well-known argument of Kervaire (see [4]) applied to the universal covering space of V, which, by hypothesis, contains a connected covering of $f(M)$.

PROPOSITION I.4. *Let $f\colon M^m \to V^{m+2}$ be an imbedding of compact manifolds that induces an isomorphism of fundamental groups and a surjection of second homotopy groups and has normal bundle ξ, with associated unit circle bundle S.*

Then the inclusion of S in $V - f(M)$ induces

(1) *an isomorphism* $H_1(S; \mathbf{Z}\pi_1(M)) \to H_1(V - f(M); \mathbf{Z}\pi_1(V))$ *and*
(2) *a surjection* $H_2(S; \mathbf{Z}\pi_1(M) \to H_2(V - f(M); \mathbf{Z}\pi_1(V))$.

Proof. If T is the total space of the unit disk bundle associated to ξ, the map of the long exact sequence of the pair (T, S) to that of the pair (V, E) $(E = \overline{V - T})$ induced by inclusion, excision, the Thom isomorphism for ξ, and the 5-lemma together imply the result.

PROPOSITION I.5. *Let f, M, V be as in* I.4 *and let S and E be the boundary and the closed complement of a tubular neighborhood of $f(M)$ in V, respectively. If K is the meridian subgroup of $G = \pi_1(E)$, then $G/[K, K]$ is isomorphic to G_f and the projection to the quotient j: $G \to G_f$ is split by the homomorphism j_s: $G_f \to G$ induced by the inclusion of S in E (see the remarks following* I.2). *Furthermore, $K = j^{-1}(C_f)$ is the normal closure of $j_s(C_f)$ and $H_2(K, \mathbf{Z}) = 0$.*

Proof. We begin by defining j_s as the composite of an isomorphism of G_f with $\pi_1(S)$ that carries C_f to the cyclic subgroup generated by a fiber with the homomorphism $\pi_1(S) \to \pi_1(E)$ induced by inclusion. Clearly, $j_s(C_f)$ will be a cyclic subgroup of G generated by a meridian class. Consider the following commutative exact diagram, induced by the composite of j_s with the projection $G \to G/[K, K]$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_f & \longrightarrow & G_f & \longrightarrow & \pi_1(M) & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow q & & \downarrow r & & \\ 0 & \longrightarrow & K/[K, K] & \longrightarrow & G/[K, K] & \longrightarrow & \pi_1(M) & \longrightarrow & 0 \end{array}$$

where r is an isomorphism. If we identify C_f with $H_1(S; \mathbf{Z}\pi_1(M))$ (i.e., the cyclic subgroup generated by a fiber) and $K/[K, K]$ with $H_1(K, \mathbf{Z}) = H_1(E; \mathbf{Z}\pi_1(M))$, then p coincides with the map induced by inclusion so that, by I.4, it must be an isomorphism. It follows, by the 5-lemma, that q is an isomorphism and we can define j as the composite of q^{-1} with the projection $G \to G/[K, K]$. It is clear that, with this definition, $j^{-1}(C_f) = K$ and therefore, by I.3, is the normal closure within itself of $j_s(C_f)$. The remaining statement follows upon considering the diagram

$$\begin{array}{ccc} H_2(S; \mathbf{Z}\pi_1(M)) & \xrightarrow{i} & H_2(E; \mathbf{Z}\pi_1(V)) \\ & \searrow^{r} & \downarrow s \\ & & H_2(G; \mathbf{Z}\pi_1(V)) \end{array}$$

where i is induced by inclusion, s by the characteristic map of E, and r by that of S. Shapiro's lemma implies that $H_2(G; \mathbf{Z}\pi_1(M)) = H_2(K; \mathbf{Z})$ and $H_2(S; \mathbf{Z}\pi_1(M)) = H_2(C_f) = 0$. The conclusion now follows from the fact that i, s, and r are surjective and r factors through $H_2(C_f) = 0$.

The following proposition is essentially the same as the theorem in the appendix of [3]. We will give a proof for the sake of completeness.

PROPOSITION I.6. *Let f, M, V, E, and S be as in* I.5 *and suppose that the dimension of M is* ≥ 3. *Then f is concordant to an imbedding f' such that* $\pi_1(E') = G_f$ *with meridian subgroup* C_f, *where E' is the closed complement of a tubular neighborhood of* $f'(M)$ *in V.*

Remark. Note that concordant imbeddings are homotopic so that $(G_{f'}, C_{f'}) = (G_f, C_f)$.

Proof. Let $G = \pi_1(E)$ and let K be the meridian subgroup. Then since G and G_f are finitely presented groups and since, by I.4, $G/[K, K] = G_f$ it follows that $[K, K]$ is normally generated in G by a finite number of elements. Attach 2-cells to E via maps representing normal generators of $[K, K]$ forming E_1. Then $H_i(E_1, E; \mathbf{Z}\pi_1(V))$ is 0 for $i \neq 2$ and a free module F for $i = 2$, and

$$H_i(E_1; \mathbf{Z}\pi_1(V)) = H_i(E; \mathbf{Z}\pi_1(V))$$

for $i \neq 2$ and $H_2(E_1; \mathbf{Z}\pi_1(V)) = H_2(E; \mathbf{Z}\pi_1(V)) \oplus F$. The universal covering space spectral sequence shows that

$$H_2(\pi_1(E_1); \mathbf{Z}\pi_1(V)) = H_2(G_f; \mathbf{Z}\pi_1(V)) = H_2(C_f)$$

(the last equality is Shapiro's lemma) is the cokernel of the Hurewicz homomorphism. It follows that we can attach 3-cells to E_1 forming E_2 so that the inclusion $E \to E_2$ is a simple $\mathbf{Z}\pi_1(V)$-homology equivalence. By an argument identical to that used in the proof of Lemma 4.3 of [2] we can perform surgery on E to obtain $E' \to E_2$ such that the new map induces an isomorphism of fundamental groups and the trace of the surgeries is a $\mathbf{Z}\pi_1(V)$-homology s-cobordism. It follows that the complement E' is the complement of a tubular neighborhood of an imbedding of M in V concordant to f and such that the fundamental group of the complement is G_f.

II. Proof of Theorem 2

The necessity of the conditions in the statement of Theorem 2 has already been proved in I.5 except for the requirement that G be finitely presented. This follows from the fact that the complement of an imbedding of compact manifolds has the homotopy type of a finite complex. It only remains to prove that the conditions are sufficient. In view of I.6 we may assume, without loss of generality, that the map f in the statement of Theorem 2 has the property that $\pi_1(V - f(M)) = G_f$ with meridian subgroup C_f. Let T be a tubular neighborhood of $f(M)$ in V and let $E = \overline{V - T}$ and $S = \partial T$. Then $\pi_1(S) = G_f$ and inclusion of S in E induces an isomorphism of fundamental group.

Suppose G is a group that satisfies the hypotheses of the theorem. We must construct a locally-flat imbedding $f'\colon M \to V$ that is concordant to f, such that $\pi_1(V - f'(M)) = G$. The splitting of j gives an injection $j_s\colon G_f \to G$.

Since G is finitely presented we can attach a wedge of circles to E (off S) and a finite number of 2-disks forming E_1 such that $\pi_1(E_1) = G$ and such that the inclusion $E \hookrightarrow E_1$ induces j_s on fundamental groups. Since j_s *splits* j and j induces an *isomorphism* (by Remark 2 following Theorem 2)

$$H_1(S; \mathbf{Z}\pi_1(V)) = H_1(E; \mathbf{Z}\pi_1(V)) \leftarrow H_1(G; \mathbf{Z}\pi_1(V)),$$

it follows that j_s induces an isomorphism

$$H_1(E; \mathbf{Z}\pi_1(V)) \rightarrow H_1(E_1; \mathbf{Z}\pi_1(V)).$$

This implies, since only 1- and 2-cells have been attached, that

$$H_i(E_1, E; \mathbf{Z}\pi_1(V)) = \begin{cases} 0, & i \neq 2 \\ F, & i = 2 \end{cases}$$

and

$$H_2(E_1; \mathbf{Z}\pi_1(V)) = H_2(E; \mathbf{Z}\pi_1(V)) \oplus F$$

where F is a free $\mathbf{Z}\pi_1(V)$-module. Since $H_2(G; \mathbf{Z}\pi_1(V)) = 0$ (by hypothesis, and by Shapiro's lemma), a variation of Hopf's theorem implies that the map $\pi_2(E_1) \rightarrow H_2(E_1 \mathbf{Z}\pi_1(V))$ induced by the Hurewicz homomorphism, is *surjective*. Then we may attach 3-cells (via maps representing *basis* elements of F) to obtain E_2 such that the inclusion i: $E \rightarrow E_2$ induces a *simple* $\mathbf{Z}\pi_1(V)$-*homology equivalence*. By an argument identical to that used in Lemma 4.3 of [2] we get a $\mathbf{Z}\pi_1(V)$-homology *s-cobordism* I: $(W; E, E') \rightarrow E_2$ such that $\tilde{I}|E = i$ and $\tilde{I}|E'$ is a 2-*connected* simple $\mathbf{Z}\pi_1(V)$-homology equivalence. Since ker $G \rightarrow \pi_1(V)$ is the normal closure (within itself) of $j_s(C_f)$, it follows (by van Kampen's theorem) that if we take the union of W with $T \times I$ along $S \times I \subset W$ we get a homotopy equivalence (i.e., $\pi_1(W \cup T \times I) = \pi_1(V)$), and the union is an *s-cobordism*).

This implies that $E' \cup_S T$ is homeomorphic to V and the imbedding $M \hookrightarrow T \hookrightarrow E' \cup_S T \hookrightarrow V$ has $\pi_1(V - \text{im}\ (M)) = G$ and is concordant to f.

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RICE UNIVERSITY
HOUSTON, TEXAS