Steenrod Coalgebras

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Citation

Statement

Simplicial abelian groups

Universal Steenrod coalgebra

Future work

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Introduction

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$$\xi_X \colon \mathrm{R}S_2 \otimes C(X) \to C(X) \otimes C(X)$$

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  and that this extends the Alexander-Whitney diagonal.
- In his construction of cup-i products, Steenrod defined a dual of this map.

We can make this *functorial* too, so any simplicial map

$$f\colon X \to Y$$

induces a commutative diagram

$$\begin{array}{ccc} \mathrm{R}S_2 \otimes C(X) & \xrightarrow{1 \otimes C(f)} & \mathrm{R}S_2 \otimes C(Y) \\ \downarrow \xi_X & & \downarrow \xi_Y \\ C(X) \otimes C(X) & \xrightarrow[C(f) \otimes C(f)]{} & C(Y) \otimes C(Y) \end{array}$$

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- What can we say about X and Y?

It turns out (with *many* qualifications) that such a chain-map induces a simplicial map

$$g_\infty \colon \mathbb{Z}_\infty X \to \mathbb{Z}_\infty Y$$

of $\mathbb{Z}$-completions "strongly related to g"

More precise statement

In (7), I prove that if X and Y are pointed, reduced, degeneracy-free simplicial sets and $g: N(X) \to N(Y)$ is a chain-map of *normalized* chain-complexes that preserves the Steenrod diagonals, then there exists a simplicial map

$$g_\infty: \mathbb{Z}_\infty X \to \mathbb{Z}_\infty Y$$

that makes the diagram

$$\begin{array}{ccc} X & & Y \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} \\ \mathbb{Z}_\infty X & \xrightarrow{g_\infty} & \mathbb{Z}_\infty Y \\ \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} \\ \tilde{\mathbb{Z}} X & \xrightarrow[\tilde{\Gamma} g]{} & \tilde{\mathbb{Z}} Y \end{array}$$

Here:

- ϕ_X, ϕ_Y are natural maps and (if the spaces X and Y are $\mathbb{Z}$*-good*) *integral homology equivalences*. This happens if X and Y are nilpotent, for instance.

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- ϕ_X, ϕ_Y are natural maps and (if the spaces X and Y are $\mathbb{Z}$*-good*) *integral homology equivalences*. This happens if X and Y are nilpotent, for instance.
- $\tilde{\Gamma} g$ is the induced map of pointed versions of the Dold-Kan functor
- If g is a homology equivalence, so is g_∞.
- The work ([3]) of Rourke and Sanderson shows that all simplicial sets are canonically homotopy equivalent to degeneracy-free ones.

Consequences

- If g is a homology equivalence and X and Y are nilpotent, then they are homotopy equivalent.

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- If g is a homology equivalence and X and Y are nilpotent, then they are homotopy equivalent.
- The Steenrod diagonal, originally used to define Steenrod *squares*, actually determines *all* Steenrod operations.

Simple example

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- Let $\Delta_2 = h([(1,2)] \otimes *)\colon C(X) \to C(X) \otimes C(X)$.

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- Let $\Delta_2 = h([(1,2)] \otimes *)\colon C(X) \to C(X) \otimes C(X)$.
- Then

$$\begin{aligned}
\partial\{(1 \otimes \Delta) \circ \Delta_2\} &= (1 \otimes \Delta) \circ \partial\Delta_2 \\
&= (1 \otimes \Delta) \circ \{(1,2) - 1\}\Delta \\
&= (1,2,3)(\Delta \otimes 1) \circ \Delta - (1 \otimes \Delta) \circ \Delta \\
&= \{(1,2,3) - 1\}(1 \otimes \Delta) \circ \Delta
\end{aligned}$$

History

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- Smirnov's proof was somewhat opaque and the community still has not assimilated it.
- Although some even questioned the result's validity, the work discussed here appears to vindicate it.

History

- In (6), the author showed that the chain-complex of a space was naturally a coalgebra over an E_∞-operad $\mathfrak{S}$ and that this could be used to iterate the cobar construction (in a paper that was also opaque and unassimilated).

History

- In (6), the author showed that the chain-complex of a space was naturally a coalgebra over an E_∞-operad $\mathfrak{S}$ and that this could be used to iterate the cobar construction (in a paper that was also opaque and unassimilated).
- The paper (5) applied those results to show that this $\mathfrak{S}$-coalgebra determined the integral homotopy type of a simply-connected space.

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- In (1)[1], Mandell showed that the mod-p cochain complex of a p-nilpotent space had a algebra structure over an operad that determined the space's p-type.

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- In (2), Mandell showed that the cochains of a nilpotent space whose homotopy groups are all *finite* have an algebra structure over an operad that determined its integral homotopy type.

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- Let **sAB** denote the category of simplicial abelian groups and **Ch**, that of chain complexes. We have inverse functors

$$\hat{N}\colon \mathbf{sAB} \to \mathbf{Ch}$$
$$\Gamma\colon \mathbf{Ch} \to \mathbf{sAB}$$

$\hat{N}(A) = A/D(A)$ — the subgroup generated by degenerate simplices and

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- the *Dold-Kan functor:*

$$\Gamma C_n = \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{m}} C_m$$

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$$\mathbb{Z}X \to \prod_{n\geq 0} K(H_n(X), n)$$

- and the *Hurewicz* map

$$\begin{aligned} h_X\colon X &\to \mathbb{Z}X \\ x &\mapsto 1\cdot x \end{aligned}$$

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The Hurewicz map is so-named because it induces the *Hurewicz homomorphism* in homotopy groups

$$\pi_n(h_X)\colon \pi_n(X) \to \pi_n(\mathbb{Z}X) = H_n(X, \mathbb{Z})$$

It is used to define the cosimplicial $\mathbb{Z}$-resolution, $\mathbb{Z}^\bullet X$, of X:

$$\tilde{\mathbb{Z}}X \overset{\partial^i}{\rightrightarrows} \tilde{\mathbb{Z}}^2 X \to \cdots$$

where the coface maps are defined by $\partial^i = \tilde{\mathbb{Z}}^{n-i+1} \circ h^*_{\tilde{\mathbb{Z}}^i X}$, $i = 0, \dots, n$.

- The total space is given by

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- where $\Delta^\bullet$ is the standard cosimplex and the hom is the set of simplicial maps commuting with all cofaces and codegeneracies

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- If X is degeneracy free, $\mathbb{Z}_\infty X$ is *determined* by $\tilde{\mathbb{Z}}X$ or $N(X)$ and
- the chain-map induced by the Hurewicz map

$$N(h)\colon N(X) \to N(\tilde{\mathbb{Z}}X)$$

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- If C is a chain complex and T is a binary tree, define $C(T)$ recursively by

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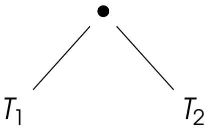

then $C(T) = \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, C(T_1) \otimes C(T_2))$

Construction

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- There exists a homomorphism
$$\xi \colon K \to \prod_{T_1, T_2} \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, C(T_1) \otimes C(T_2))$$

Construction

Given the diagram

$$\begin{array}{ccc} & & \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, K \otimes K) \\ & & \downarrow \gamma \\ K & \xrightarrow{\xi} & \prod_{T_1,T_2} \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, C(T_1) \otimes C(T_2)) \end{array}$$

define

- $U_0 = K$

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define

- $U_0 = K$
- $U_{i+1} = \xi^{-1}\left(\gamma\left(\mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, U_i \otimes U_i)\right)\right) \subset U_i$

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- The map
$$\gamma^{-1} \circ \xi \colon L_{\mathscr{F}}(C) \to \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, L_{\mathscr{F}}(C) \otimes L_{\mathscr{F}}(C))$$
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- this is equipped with a chain-map

$$\epsilon_C \colon L_{\mathscr{F}}(C) \to C$$

  called its *cogeneration-map*

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- Suppose C is a Steenrod coalgebra with underlying chain-complex C and structure-map

$$\alpha\colon C \to \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, C \otimes C)$$

For each binary tree T there exists a chain-map

$$f(T)\colon C \to C(T)$$

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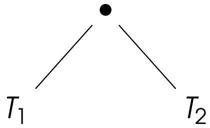

we define $f(T) = \mathrm{Hom}_{\mathbb{Z}}(1, f(T_1) \otimes f(T_2)) \circ \alpha$

The $\{f(T)\}$ induce a map

$$\beta = 1 \oplus \prod_T f(T) \colon C \hookrightarrow K$$

The commutativity of

$$\begin{array}{ccc}
 & & \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, C \otimes C) \\
 & \overset{\alpha}{\nearrow} & \Big\downarrow \mathrm{Hom}_{\mathbb{Z}}(1, \beta \otimes \beta) \\
C & & \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, K \otimes K) \\
\Big\downarrow \beta & & \Big\downarrow \xi \\
K & \xrightarrow{\gamma} & \prod_{T_1, T_2} \mathrm{Hom}_{\mathbb{Z}S_2}(\mathrm{R}S_2, C(T_1) \otimes C(T_2))
\end{array}$$

shows that $\beta(C) \subset L_{\mathscr{F}}(C) \subset K$ and that β is a morphism of Steenrod coalgebras.

Universal Property

If C is a Steenrod coalgebra and $f\colon C \to D$ is a chain-map, then there exists a *unique* Steenrod-coalgebra morphism

$$\bar{f} = L_{\mathscr{F}}(f) \circ \beta\colon C \to L_{\mathscr{F}}(D)$$

that makes the diagram

$$\begin{array}{ccc} C & \xrightarrow{\bar{f}} & L_{\mathscr{F}}(D) \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \epsilon_D} \\ & & D \end{array}$$

commute.

- Define a chain-map

$$N(\tilde{\mathbb{Z}}X) \to C(X)$$
$$1 \cdot \left(\sum_i \alpha_i \sigma_i\right) \mapsto \sum_i \alpha_i \sigma_i$$

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- Since X has degenerate simplices, there is no chain-map $N(\tilde{\mathbb{Z}}X) \to N(X)$ that is *injective on simplices*, but there *is* one to $C(X)$.

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- This induces a unique coalgebra-morphism

$$\gamma: N(\tilde{\mathbb{Z}}X) \to L_{\mathscr{F}}(C(X))$$

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Incidentally, this is the reason we need X to be degeneracy-free.

If h is the Hurewicz map, the *uniqueness* of those morphisms implies that the diagram

$$\begin{array}{ccc} N(X) & \xrightarrow{\beta} & L_{\mathscr{F}}(C(X)) \\ & \underset{N(h)}{\searrow} & \uparrow \gamma \\ & & N(\tilde{\mathbb{Z}}X) \end{array}$$

commutes.

Punch line

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- It follows that $N(h)$ (and $\mathbb{Z}_\infty X$) is *determined* by the Steenrod coalgebra structure of $N(X)$

If Δ^n is an n-simplex, Steenrod showed that

$$\xi_{\Delta^n}(e_n \otimes [\Delta^n]) = \pm[\Delta^n] \otimes [\Delta^n]$$

where $e_n \in (\mathrm{R}S_2)_n$ is the generator and $[\Delta^n]$ is the element of $N(\Delta^n)_n$ generated by Δ^n.

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If X is a simplicial set and we map the Steenrod coalgebra of $\tilde{\mathbb{Z}}X$ to $L_{\mathscr{F}}(C(X))$, we get a diagram

$$N(\tilde{\mathbb{Z}}X) \to L_{\mathscr{F}}(C(X)) \to \prod_{k\geq 1} C(X)^{\otimes k}$$

where a simplex $c \in \tilde{\mathbb{Z}}X_n$ maps to

$$\{c, c \otimes c, c \otimes c \otimes c, \dots\}$$

when evaluated on $\{e_n, e_n \circ_1 e_n, \dots\}$.

If $\{c_1, \dots, c_k\} \in \tilde{\mathbb{Z}}X$ are distinct elements, it is not hard to see that their images under the map

$$N(\tilde{\mathbb{Z}}X) \hookrightarrow N(\tilde{\mathbb{Z}}X) \otimes \mathbb{Q} \to L_{\mathscr{F}}(C(X)) \otimes \mathbb{Q} \to \prod_{k \geq 1} C(X)^{\otimes k} \otimes \mathbb{Q}$$

are linearly independent.

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- Must have an action of the Eccles-Barratt operad, $\mathfrak{S}$.
- One can define a *cellular* Steenrod coalgebra as one in which the image of the classifying map

$$\alpha\colon C \to L_{\mathcal{F}} C \to L_{\mathscr{F}}(\{\tilde{\Gamma} C\})$$

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- Such Steenrod coalgebras have a *"Hurewicz map"*

$$h\colon C \to N(\tilde{\Gamma} C)$$

One can use this to construct a *Hurewicz realization* of C — a cosimplicial space

$$\tilde{\Gamma} C \overset{\tilde{\Gamma} h_i}{\rightrightarrows} \tilde{\mathbb{Z}}\tilde{\Gamma} C \overset{\tilde{\mathbb{Z}}\tilde{\Gamma} h_i}{\rightrightarrows} \tilde{\mathbb{Z}}^2\tilde{\Gamma} C \rightrightarrows \cdots$$

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